# Geometric Concepts in Ordered Algebra

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# Introduction

We study the concept of "distance" and its dependence on the concept of "order " in algebraic systems.For this ,we study "Distance Spaces" where in the distances are either non-negative real numbers or elements of Boolean algebras or elements of lattices ordered groups or more generally elements of an ordered algebra.

Basically any distance space needs a nonempty set X and a set D of distances and a distance function $d : X \times X \to D$ i.e for any $x, y \in X, d(x, y)$ is an element of D which is called the distance from x to y. Usually the distance set needs an elemtn :zero" and an order $\leq$and a binary operation + besides a distance function d to add distances, to compare distances and to separate points via distance. The classical example of a distance space (or a metric space) is the metric space $(X, \mathbb{R}^+, d)$ where X is any nonempty set , $\mathbb{R}^+$ is the set of all non-negative real numbers and the distance $d(x, y)$ from x to y of X is a non-negative real number.Here in $\mathbb{R}^+$ the order $\leq$ and + have usual meanings as in real number system.

Also if X is the set of points on the Eucledian plance i.e., $X = \mathbb{R} \times \mathbb{R}$ then $d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ is the most familiar example of a metric space. Here also the distance set is $\mathbb{R}^+$and $\leq$,+ in $\mathbb{R}^+$ have usual meanings as in $\mathbb{R}$.Later Blumenthal and followers of his thought such as David Ellis have developed the study of Boolean metrix spaces i.e., the distance is an element of a Boolean Algebra. Normed Boolean Vector spaces introduced and studied by Subrahmanyam, and the geometric studies of p-rings and commutative regular rings with unity are all studies of Boolean metric spaces. The concept of"autometrized space" also arose at the same time meaning the distances are elements of the same space X itself. Real line ($\mathbb{R}^+$)with distance from x to y as the absolute value $|x - y|$, is the oldest autometrized space. Boolean Algebras, Browerian Algebras, Semi-Browerian algebras with symmetric difference as a distance function and commutative lattice ordered groups with $|x - y|$as the distance from x to y and MV algebras with distances as $x\overline{y} + y\overline{x}$ as $d(x, y)$ are all some examples of autometrized spaces besides their generalisations such as Drl-monoids.

Also like Boolean metric spaces, the study of distance spaces with distances in abelian lattice ordered groups, spaces with distance in a value semi group and spaces with distance in auto metrized Algebras (A-metric spaces) all come under the study of Distance spaces. All these spaces may be broadly called as generalised metric spaces. The contribution of metric space to Mathematics in general and to topology and Functional Analysis in particular are significant. For instance any metric space gives rise to a topology and it is well known that all topologies do not arise from metric spaces.But the study of Ralph Kopperman shows that all topologies arise from continuity spaces via genralised metrics, more specifically via distance spaces with distance in a value semi group.

Metric spaces and the above genralized metric spaces all will form OMM-spaces(Ordered monoid metric spaces) which notion we introduce in this book.

Our stress in this book is on the study of concepts of betweenness, segments, congruences, isometries in general OMM-spaces and in some special cases of OMM-spaces.

# Chapter 1

# Ordered Monoid Metrized spaces

In this chapter we introduce the notion of an "ordered monoid metrized space" (briefly OMM space) which includes the notion of metric spaces, Boolean metric spaces, $G$-metric spaces (where $G$ is abelian lattice ordered group), Autometrized algberas, continuity spaces and establish a few general properties, besides proving that any topology arises from a generalised metric space, the result of Ralph Kopperman.

## 1 OMM Spaces

### 1.1 Definition and Examples

**Definition 1.1.** *Let $X$ be a nonempty set, $A = (A, +, 0, \leq)$ be a commutative ordered (partially ordered) monoid, where by we mean $(A, +, 0)$ is a commutative monoid with $0$ as additive identity, and $\leq$ is a reflextive, antisymmetric and transitive ordering on $A$ such that $x \leq y$ in $A$ implies $a + x < a + y$ for all $a$ in $A$; and $d : X \times X \to A$ is a mapping such that*

$(m_1)$ *$d(x, y) \geq 0$ for all $x, y \in X$, $d(x, x) = 0$ for all $x, y \in A$.*

$(m_2)$ *$d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z$ in $A$. Then $(X, A, d)$ is called an OMM -space and $d$ is called a distance function.*
*If an OMM-space $(X, A, d)$ satisfies*

$(m_3)$ *$d(x, y) = d(y, x)$ for all $x, y$ in $X$, then, $(X, A, d)$ is called a symmetric space. Also it is called a pseudo metric OMM space if $(X, A, d)$ satisfies*

$(m_4)$ *$d(x, y) = 0$ implies $x = y$ for any $x, y$ in $X$; then, $(X, A, d)$ is said to separate points.*
*If $(X, A, d)$ satisfies $(m_1)$ through $(m_4)$, then $(X, A, d)$ is called a metric OMM-space.*

**Note.** Here, we have assumed $\leq$ to be a partial ordering, but in some cases for instance in Newman algberas and in Griss algebras the inclusion $\leq$ is not transitive but in large number of examples $\leq$ is transitive.
The classical example of usual metric space $(X, \mathbb{R}, d)$ where $\mathbb{R}$ is the real number system and $d : X \times X \to \mathbb{R}$

is a metric,i.e., $d$ satisfying all $(m_1)$ through $(m_4)$. In particular, $(\mathbb{R}, \mathbb{R}, d)$ where $d(x, y) = |x - y|$ is a metric OMM-space.

**Definition 1.2** (Autometrized Algebra). *A system $(A, +, \leq, 0, *)$ is called an automertized algebra if*

*(i)* $(A, +, 0)$ *is a commutative monoid with* 0 *as identity.*

*(ii)* $\leq$ *is a reflextive, antisymmetric ordering on $A$.*

*(ii)* $* : A \times A \to A$ *is a mapping satisfying* $(m_1)$ *and* $(m_2)$.
*If we assume $\leq$ is a partial ordering with $x \leq y$ implying $a + x \leq a + y$, then, $(A, A, *)$ is an OMM-space.*

**Definition 1.3** (Lattice Ordered Autometrized Algbera). *A system $A = (A, +, 0, \cup, \cap, *)$ where*

*(i)* $(A, +, 0)$ *is a commutative monoid with* 0 *as additive identity.*

*(ii)* $(A, \cup, \cap)$ *is a lattice (which induce the partial ordering $\leq$) with $x \leq y$ implying $a + x \leq a + y$.*

*(iii)* $*$ *satisfies* $(m_1)$, $(m_2)$, $(m_3)$ *and* $(m_4)$. *Then $(A, +, \cup, \cap, *)$ is called a lattice ordered autometrized algebra.*

**Remark 1.1.** *Clearly, any latticed ordered autometrized algbera is a metric OMM-space $(A, A, *)$. Boolean Algebras, Brouwerian algebras, MV algebras, commutative lattice ordered autometrized algebras,and $DR\ell$-monoids are all lattice ordered autometrized algebras.*

**Definition 1.4** (Boolean metric space). *Let $X$ be a non empty set and let $G = (G, +, \leq, 0, -)$ be a commutative lattice ordered group (lattice operations induced by $\leq$ are denoted by $\cup$ and $\cap$) and $d$ is a mapping from $X \times X \to G$, satisfying $(m_1)$ through $(m_4)$. Then, $(X, G, d)$ is called a $G$-metric space.*

Clearly, any $G$-metric space is an OMM -space and the usual metric space $(X, \mathbb{R}, d)$ is a $G$-metric space.

**Definition 1.5** (Value Semigroup). *Let $A = (A, +, 0, \infty, \cap)$ be a system, where*

*(i)* $(A, +, 0, \cap)$ *is a meet semilattice,* $(A, +, 0)$ *is a monoid with $a + (b \cap e) = (a + b) \cap (a + e)$.*

*(ii)* *$a \leq b$ implies there exists $x$ in $A$ such that $a + x = b$.*

*(iii)* *$0 \neq \infty$, $0 \leq x \leq \infty$ for any $x$ in $A$, be in $x + \infty = \infty$.*

*(iv)* *To each $a$ in $A$, there exists a unique $b$ in $A$ such that $b(= \frac{a}{2})$, $b + b = a$.*

Note that in usual ordered set terminology, the element $\infty$ is denoted by 1, but to avoid confusion when we are dealing with subsets of $\mathbb{R}$ and direct products, we use the symbol $\infty$. Note also that the inclusion $\leq$ is a partial ordering.

**Definition 1.6** (Positives). *Let $A$ be a value semigroup. A set of positives $P \subset A$ is defined by*

*(i)* *$r, s \in P$ implies $r \cap s \in P$.*

*(ii)* *$r \in P$, $r \leq a$ implies $a \in P$.*

*(iii)* *$r \in P$ implies $\frac{r}{2} \in P$.*

*(iv)* $a \leq b + r$ *for each* $r \in P$ *implies* $a \leq b$.

**Theorem 1.1.** *Let* $\{A_i | i \in I\}$ *be a non empty family of value semigroups with* $P_i$ *as a set of positives for each* $i \in I$.
*Let* $A = \prod_{i \in I} A_i$, $P = \{r \in A | r_i \in P_i \text{ with } r_i = \infty, \text{for all but a finite nuber of } i \in I\}$. *Then* $A$ *is a value semigroup with* $P$ *as a set of positives.*

*Proof.* (i) Here for $r, s$ in $A$, $r + s$ is defined component wise, i.e., $(r + s)(i) = r(i) + s(i)$ for each $i \in I$ (We also denote this as $(r + s)_i = r_i + s_i$ when there is no confusion), $0 \in P$ is defined by $0(i) = 0_i$, $\infty \in A$ is defined by $\infty(i) = \infty_i$ and also $r \leq s$ if and only if $r(i) \leq s(i)$ for all $i \in I$.

(ii) For $r, s$ in $P$, $(r \cap s)_i = r(i) \cap s(i)$ is finite for only a finite number of indices $i$, i.e., for all other "$i$"s is $\infty_i$.

(iii) If $r \in P$, $s \in A$, $r \leq s$, then, $s_{er} = \infty_i$ for all but a finite number of indices $i$.

(iv) If $r \in P$, then $\frac{r}{2} \in P$ is also clear.

(v) Let $a \leq b + r$ for all $r \in P$. We need to show that $a \leq b$, i.e., $a(i) \leq b(i)$ for all $i \in I$.
For some $j \in I$ and let $s_j$ be any arbitrary element of $P_j$, define $q$ in $A$ by $q(0) = \infty_i$ for $i \neq j$ and $q(j) = s_j$. Clearly $q \in P$. So by hypothesis, $a \leq b + q$
$\implies a_j \leq b_j + s_j$
$\implies a_j \leq b_j$ as $s_j$ is arbitrary in $P_j$
$\implies a \leq b$ as $j$ is also arbitrary.
Hence, $P$ is a set of positives for $A$.

□

**Definition 1.7** (Continuity space)**.** *A continuity space* $(X, d, A, P)$ *is a system where* $X$ *is a non empty set,* $A$ *is a value semigroup* $A = (A, +, 0, \infty)$, $P$ *a set of positives for* $A$, *and* $d : X \times X \to A$ *is a mapping satisfying* $(m_1)$ *and* $(m_2)$.
*Note that* $(X, A, d)$ *is an OMM-space.*

**Definition 1.8.** *Let* $(X, d, A, P)$ *be a continuity space. Let* $x \in X$ *and* $b \in A$. *Then, the set* $N_b(x) = \{y \in X | d(x, y) \leq b\}$ *is called the closed ball of radius* $b$ *about* $x$, *and let*

$$TO(X) = \{S \subseteq X | \ if \ x \in S, then \ N_r(x) \subset S \ for \ some \ r \ in P\}.$$

**Theorem 1.2.** $TO(X)$ *is a topology on* $X$. *(we call this as the topology arising from* $X$*)*

*Proof.* $\emptyset, X \in TO(X)$.
Arbitrary union of members of $TO(X)$ is clearly a member of $TO(X)$.
For a finite number (enough if we consider two members of $TO(X)$), because, $N_r(x) \cap N_s(x) = N_{x \cap s}(x)$ follows that $TO(X)$ is a topology on $X$. □

**Theorem 1.3.** *Every topology is a* $TO(X)$.

We need a lemma

**Lemma 1.1.** *Let $(X,T)$ be any topological space. Let $s \in T$ and let $R$ be the additive semigroup of extended non negative real numbers and let $P = (0,\infty]$. Then, there exists a continuity function $d_s : X \times X \to R$ such that $To(x_s) = \{\emptyset, S, X\}$ where $X_s = X$, for the continuity space $(X_S, R, d_S, P)$*

*Proof.* Fix some $q > 0$.Define $d_S : X \times X \to R$ by

$$d_S(x,y) = \begin{cases} 0 & if\ (x \in S \Rightarrow y \in S) \\ q & otherwise \end{cases}$$

Now $d(x,x) = 0$ is clear. i.e., $(m_1)$ is satisfied.
Let $x, y, z \in X_s (= X)$. If $d_s(x,y)$ or $d_s(y,z)$ is $q$, then clearly $d_s(x,z) \leq d_s(x,y) + d_s(y,z)$.
Let $d_s(x,y) = d_s(y,z) = 0$. $\Rightarrow z \in S$ whenever $x \in S$. This implies that $d_s(x,z) = 0$. Thus $(m_2)$(The triangle inequality) is satisfied. So $(X_s, R, d_s, P)$ is a continuity space.
If $p < q$, then $N_p(X) = \{y \in X_s | d_s(x,y) \leq p\}$

$$=\{\text{y—d}_s(x,y) = 0\} \quad = \{y \in X_s | (x \in S \implies y \in S)\} = \begin{cases} S & if\ x \in S \\ x_S & otherwise \end{cases}$$

If $p > q$, then, clearly $N_p(x) = x_S$. In any case, $N_p(x)$ is $S$ or $X_s$. Also $\emptyset$, $X_S \in TO(X_S)$. If $x \in S$, $0 < p < q$, then $N_p(x) \subset S$ so that $S \in To(X_S)$. If $Q \in To(X_s)$ and if $Q \neq \emptyset$, let $x \in Q \implies \exists p > 0$ such that $S \subseteq N_p(x) \subseteq Q$. Thus $S \subseteq Q$. If $S \neq Q$, we can choose $x \in Q - S$ and then $X = N_p(x) \subseteq Q$ implying $X = Q$. Thus $To(x_S) = \{\emptyset, S, X_s\} \subseteq T$. □

Proof of the theorem Let $(X,Y)$ be any topological space. Let $R = [0,\infty]$. Fix some $q \in (0,\infty]$. Let $A = \prod_{S \in Y} R_S$ where $R_S = R$ for all $S \in Y$. Let $P = \{a \in A |\ a_S = \infty\ for\ all\ but\ a\ finite\ number\ of\ SinY; and\ a_S \in (0,\infty]\ for\ all\ those\ finite\ number\ of\ open\ sets\ S\ in\ Y\}$.
Define $d : X \times X \to A$ by $d(x,y)(S) = d_s(x,y)$ for all $S \in Y (d_S(x,y)$as defined in the lemma). Now $(X,d,A,P)$ is a continuity space. It is enough to show that $To(x) = y$. Let $S \in Y$, $p \in (0,\infty]$ and define $r(S,p) \in A$ by $r(S,p)(S) = p$ and $r(S,p)(S') = \infty$ for any open set $S' \neq S$ in $Y$. Clearly, $r(S,p) \in P$.

$N_{r(s,p)} = \{y \in X | d(x,y) \leq r(s,p)\} = \begin{cases} \{y \in X | d_S(x,y) \leq p\} = S & \text{if } x \in S \text{ and also if } p < q \\ x & otherwise \end{cases}$ This shows that $y \in To(x)$ (Since if $x \in S$, then for $p < q$, $r(s,p) \in P$ and $N_{r(s,p)}(x) \subseteq S$).
To show that $To(x) \subseteq y$, But $Q \in To(x)$.
For $x \in Q$, then for some $r_x \in P$, $N_{r_x}(x) \subseteq Q$. But $\{S | r_x(S) < \infty\}$ is finite since $r_x \in P$. But $r_x(S_{\ell'}) = p_i$ for $1 \leq \ell' \leq n$ and $r_x(S) = \infty$ for $S' \notin \{S_1, S_2, \cdots, S_n\}$. Then, $r_x = r(S_1,p_1) \cap \cdots \cap r(S_n,p_n)$. Thus $N_{r_x}$ is the finite intersection of $N_r(S_i,p_i)$. Since each of these is either $x$ or $S_i$ follows that $N_{r_x}(x) \in y$. Thus $Q \subseteq \cup_{x \in Q} N_{r_x}(x) \subseteq Q$ and so $Q$ is a union of members of $y$ which implies $To(x) \subseteq y$. Thus $To(x) = y$. $To(x)$ is a $T_1$-space if and only if $x$ is separated (i.e., $d$ satisfies $(m_4)$ also)

*Proof.* Suppose $x$ is separated. If $x \in x - \{y\}$, then, $d(x,y) \neq 0$. So for some $r \in p$, $d(x,y) \leq r$. For this $r \in N_r(x) \subseteq X - \{y\}$ and since $x$ was arbitrary, $x - \{y\}$ is open, i.e., $\{y\}$ is closed.

Conversely, if $x$ is a $T_1$-space and $x, y \in x$, $x \neq y$, then, $x \in X - \{y\}$ so for $r \in P$, $N_r(x) \subset x - \{y\}$. Thus $d(r,s) \nleqq r$ and in particular, $d(x,y) \neq 0$. □

# 2 Isometries and Congruences

**Definition 2.1.** *Let $(X, A, d)$ be an OMM-space. A mapping $\sigma : x \to x$ is called a distance preserving map from $x$ to $x$ if and only if $d(x,y) = d(\sigma(x), \sigma(y))$ for all $x, y$ in $x$. $\sigma$ is called an isometry if $\sigma$ is a bijection and distance preserving. Distance preserving maps are also called Congruences on $x$.*

Note. If we assume that $d$ satisfies $(m_4)$ also, then every congruence (or distance preserving) map is also an injection.

**Theorem 2.1.** *If $(X, A, d)$ is an OMM-space, let $X^*$ denote the set of all isometries of $x$. Then under the composition of mappings, $(X^*, 0, i_X)$ is a group (where $i_X : X \to X$ is the identity mapping of $X$).*

*Proof.* $i_X : X \to X$ is an isometry of $X$. If $\sigma$ and $\tau$ are isometries of $X$, then $d((\sigma o \tau)(x), (\sigma o \tau)(y)) = d(\tau(x), \tau(y)) = d(x,y)$ so that $\sigma o \tau$ is distance preserving. Since $\sigma$ and $\tau$ are bijections, $\sigma o \tau$ is also a bijection. Also composition of mappings is associative. Also for any isometry $\sigma : X \to X$, for any $x, y$ in $x$, $x = \sigma(\sigma^{-1}(x))$, $y = \sigma(\sigma^{-1}(y))$, follows $d(x,y) = d(\sigma(\sigma^{-1}(x)), \sigma(\sigma^{-1}(y))) = d(\sigma^{-1}(x), \sigma^{-1}(y))$ so that $\sigma^{-1}$ is distance preserving. As $\sigma^{-1}$ is also a bijection, follows that $\sigma^{-1}$ is an isometry. Hence, $X^*$ is a group. □

Note: We usually write $X^*$ is a group also it is convienient to use $*$ in the usual set notation for $d$, i.e., we use $x * y$ for $\ell(x,y)$.

**Definition 2.2.** *Let $\{x_i | i \in I\}$ be a non empty family of OMM-spaces $(x_i, A_i, d_i)$ for each $i \in I$. Let $X = \prod_{i \in I} X_i$, $A = \prod_{i \in I} A_i$. In $A$ define the operations componentwise. Define $d : X \times X \to A$ by $d(x,y)(i) = d_i(p_i(x), p_i(y))$ where $p_i : X \to X_i$ is the $i^{th}$ projection map. Then $(X, A, d)$ is an OMM-space.*

Note that for $x \in X$, we many times write $X_i$ for $p_i(x)$ and say $X_i$ is the $i^{th}$ component of $X$. Also for 0(zero) in $A$, we write $o_i$ for the zero of $A_i$ (and usually we use the symbol 0 for $0_i$ as long as there is no confusion). Also note that in $A$, $a \leq b$ if and only if $p_i(a) \leq p_i(b)$, i.e., $a_i \leq b_i$ for all $i$.
Recall that in Theorem 1.3 we have contributed a family of OMM-spaces $\{(x_S, A_S, d_S) | S \in y\}$ where $x_S = y$ for all $S \in y$ and defined $d : x \times x \to \prod_{S \in y} A_S$ by $d(x,y)S = d_S(x,y)$. This is also a type of product space, but this differs from the direct product space defined above.

**Theorem 2.2.** *Let $\{(X_i, A_i, d_i) | i \in I\}$ be a non empty family of metric OMM-spaces. Let $X^*$ be the group of all isometries of $X = (\prod x_i, \prod A_i, d)$ and let $x_i^*$ be the group of all isometries of $(x_i, A_i, d_i)$ for each $i$ in $J$. Then $x^*$ is isomorphic with $\prod_{i \in I} x_i^*$.*

*Proof.* We are given that each $d_i$ satisfies $(m_1)$ through $(m_4)$. Then, clearly, $d$ also satisfies $(m_1)$ through $(m_4)$. Let $\sigma : x \to x$ be an isometry. For $d$ and $d_i$, we use in set notation $\vee$ (for both $d$ and $d_i$ and the content matter $\ell$-clear) For each $i \in I$, we define $\sigma_i : x_i \to x_i$ by $\sigma_i(x_i) = (\sigma(x))_i$ where $x$ is any element of $x$ with $p_i(x) = x_i$. Now we need to show that this definition is well defined. Let $x_i \in X_i$ and let $x, y$ be in $X$ such that $p_i(x) =$

$p_i(y) = x_i$. As $\sigma$ is an isometry, we have $x * y = \sigma(x) * \sigma(y) \implies p_i(x * y) = p_i(\sigma(x) * \sigma(y)) = (\sigma(x))_i * (\sigma(y))_i$, i.e., $x_i * y_i = (\sigma(x))_i * (\sigma(y))_i \implies 0 = (\sigma(x))_i * (\sigma(y))_i \implies (\sigma(x))_i = (\sigma(y))_i$ (by thm). This implies that $\sigma_i$ is well defined.

Now for any $x_i, y_i$ in $X_i$ (with $p_i(x) = x_i$, $p_i(y) = y_i$), $(\sigma_i(x) \vee (\sigma_i(y)) = (\sigma(x))_i \vee (\sigma(y))_i = (\sigma(x) \vee \sigma(y))_i = (x \vee y)_i = x_i \vee y_i$ so that $\sigma_i$ is distance preserving. This implies $\sigma_i$ is one-to-one by $(m_4)$.

Now we show that $\sigma_i$ is a surjection. Let $y_i \in x_i$. Choose $y$ in $x$ such that $p_i(y) = y_i$. Now as $\sigma$ is a surjection, $\exists z$ in $x$ such that $\sigma(z) = y \implies (\sigma(z))_i = y_i \implies \sigma_i(z_i) = y_i$(where $yz_i$ is the $i^{th}$ component of $z$)

$\implies \sigma_i$ is a surjection. Thus $\sigma_i$ is an isometry of $x_i$.

Conversely, $\sigma_i : x_i \to x_i$ be an isometry for each $i \in I$. Define $\sigma : x \to x$ by $\sigma(x)_i = \sigma_i(x_i)$. $\sigma$ is clearly well defined and for any $x, y$ in $x$.

$(\sigma(x) * \sigma(y))_i = (\sigma(x))_i * (\sigma(y))_i = \sigma_i(x_i) * \sigma_i(y_i) = x_i * y_i = (x * y)_i$.

$\implies \sigma$ is distance preserving. $\implies \sigma$ is also an injection by $(m_4)$.

We show that $\sigma$ is a surjection. Let $y \in x \implies y_i \in x_i \implies \exists z_i \in x_i \ni \sigma_i(z_i) = y_i$. Let $z$ be element in $x$, defined by $p_i(z) = z_i \implies \sigma(z) = y$. Thus $\sigma$ is a surjection and hence an isometry of $x$. Now the correspondence $\sigma \mapsto \sigma^*$ from $X^* \to \prod_{i \in I} x_i^*$ given by $(\sigma^*)_i = \sigma_i$ gives an isomorphism form $x^*$ to $\prod_{i \in I} x_i^*$. □

# 3 Betweenness

Whenever there is a concept of distance between points, a natural question arises of a mathematical definition of betweenness.

Pitcher and Smily [] discussed several consequences and desirable properties of the concept of "betweenness."

Penning and Subrahmeyaw [] studied the concept of betweenness in Boolean metric spaces and Boolean vector spaces.

Here we introduce the concept of (metric) betweenness in OMM-spaces and discuss only a few properties and we study more about in the foregoing in the next chapters, in other examples of special cases of OMM-spaces.

**Definition 3.1.** *Let $(X, A, d)$ be an OMM-space. For any $a, b, c$ in $X$, we say $b$ lies between $a$ and $e$ (metrically) in symbols $(a, b, c)M$ if and only if $(a * b) + (b * a) = a * c$.*

**Theorem 3.1.** *If $(X, A, d)$ is a symmetric OMM-space, then, $(a, b, c)M$ if and only if $(c, b, a)M$.*

*Proof.* Symmetric means $x * y = y * x$ for all $x$ and $y$ and so the proof is obvious. □

We say the betweenness relation has transitivity $t_1$ if:

$$(a, b, c)M \text{and} (a, d, b)M \implies (d, b, c)M$$

We say $\ell$ has transitivity $t_2$ if:

$$(a, b, c)M \text{and} (a, d, b)M \implies (a, d, c)M$$

**Theorem 3.2.** *Metric betweenness in OMM-space $(X, A, d)$ has transitivity $t_2$.*

*Proof.* Let $(a,b,c)M$ and $(a,d,b)M$
$\implies (a*b)+(b*c)=a*c$ and $(a*d)+(d*b)=a*b$
$\implies (a*b)+(d*b)+(b*c)=a*c$
$\implies (a*c)\geq(a*d)+(d*c)\geq(a*c)$ (by $m_2$)
$\implies a*c=(a*d)+(d*c)$, i.e., $(a,d,c)M$. □

**Definition 3.2.** *Let $(X,A,d)$ be a metric OMM-space. If $a,b,c$ are any distinct three elements of $X$, we say they constitute a triangle with vertices $a,b,c$ and sides $a*b, b*c, c*a$. We denote this triangle by $\delta(a,b,c)$.*

Now we have:

**Theorem 3.3.** *If in the metric OMM-space $(X,A,*)$ if the metric betweenness has transitivity $t_1$, then, $X$ is free of isosceles traingles in which the sum of any two sides is their supremum.*

*Proof.* Suppose in $(X,A,*)$, metric betweenness has transitivity $t_1$. Let $\delta(a,b,c)$ be a triangle with $a*b=a*c$ and the sum of any two sides is their supremum.
Note that $b*c\leq(b*a)+(a*c)=(b*a)\cup(a*c)=a*b(=a*c)$
$\implies (a,b,c)M$ and $(a,c,b)M$
$\implies (c,b,c)M$ by transitivity $t_1$
$0=c*c=(c*b)+(b*c)\geq b*c\geq 0$
$\implies b=c$, a contradiction. Hence, the result. □

**Definition 3.3.** *A betweenness relation $R$ is said to have special inner point property for $a,b,c$ if and only if $(a,b,c)R$ and $(a,c,b)R$ implies $b=c$.*

**Theorem 3.4.** *(i) A metric OMM-space $(X,A,*)$ has special inner point property for metric betweenness if $+$ is cancellative.*

*(ii) In the metric OMM-space $(X,A,d)$ if $(A,+,0)$ is a join semi lattice, then the special inner point property fails for $a,b,c$ if they are the vertices of an isosceles triangle.*

*Proof.* We have $+$ in a join semi lattice operation so $0\leq a$ for all $a$. Let $\delta(a,b,c)$ be a triangle such that $a*b=a*c$. Now, $(a*b)+(b*c)\leq(a*b)+(b*c)+(c*a)=(a*b)+(a*c)=a*b$
$\implies (a,b,c)M$ holds.
Also $(a*c)+(c*b)\leq(a*c)+(c*b)+(a*b)=(a*c)+(a*b)=a*b \implies (a,c,b)M$. But $b\neq c$. Hence, the special inner point property fails for $a,b,c$. □

Note: In the above theorem, mislead of assuming the join semi lattice operation, we may have that for triangle $\delta(a,b,c)$ the sum of any two sides is their supremum.

# 4 Autometrized Algebras

An autometrized algebra is a system $(A,+,\leq,*,0)$, where

(i) $(A, +, \leq, 0)$ is a commutative partially ordered monoid (with 0 as identity and $a \leq b \implies x + a \leq x + b$)

(ii) $* : A \times A \to A$ is a mapping satisfying $(m_1)$ through $(m_4)$.

**Example 4.1.** *(i)* $(\mathbb{R}, +, 0, \leq, *)$ *where* $\mathbb{R}$ *is the real number system,* $+$, $0$, $\leq$ *have usual meanings as in* $\mathbb{R}$, *and* $a * b = |a - b|$ *where* $|x|$ *is the absolute value of* $x$. *This is the most classical example.*
*Also* $\mathbb{Q}$, $\mathbb{Z}$ *with the usual meanings of* $+, 0, -$ *are also autometrized algebra.*

*(ii)* $\mathbb{R} \times \mathbb{R}$ *with* $+, \leq, 0$ *defined component-wise and with* $d((a_1, a_2), (b_1, b_2)) = (|a_1 - b_1|, |a_2 - b_2|)$ *is also an autometrized algbera.*

*(iii) As a trivial example, let* $(A, +, \leq, 0)$ *be a commutative partially ordered monoid and let* $d > 0 (d \neq 0$, $\alpha \geq 0)$ *be an element of* $A$.

*(iv) Definition of subautometrized algebra*
*Let* $(A, +, \leq, 0, *)$ *be an autometrized algebra and let* $S$ *be a non empty subset of* $A$ *such that* $0 \in S$, $a * b$, $a + b \in S$ *whenever* $a, b$ *are in* $S$.
*For* $x, y$ *in* $S$, $x \leq y$ *has the usual meaning as in* $A$. *Then* $S$ *in called a sub-autometrized algebra of* $A$.
*In example* $(i)$, $\mathbb{Q}, \mathbb{Z}$ *are subautometrized algebra of* $\mathbb{R}$.

**Definition 4.1.** *Let* $\{A_i | i \in I\}$ *be a non empty family of autometrized algebras* $(A_i, +, \leq, 0, *)$. *Then* $\prod_{i \in I} A_i$ *is an autometrized algebra with* $+, \leq, *, 0$ *defined as componentwise (as Definition 2.2 for OMM-spaces).*
*Note that* $(m_1)$ *through* $(m_4)$ *are satisfied in* $\prod A_i$ *as hold in each* $A_i$.

**Definition 4.2.** *Let* $(A, +, \leq, 0, *)$ *be an autometrized algebra. We say it is regular if* $a * 0 = a$ *for all* $a$ *in* $A$.
*We also say,* $*$ *is a regular metric on* $A$.
*As example, Boolean Algebras, Brouwerian algberas, MU algbebras are all regular autometrized algberas with symmmetric difference as distance.*

Now we have a small but very interesting theorem on the isometries of a regular autometrized algebra, which has many applications.

**Theorem 4.1.** *Let* $(A, +, \leq, 0, *)$ *be an autometrized Algbera. Then every isometry* $\sigma$ *of* $A$ *is of the form* $x \mapsto a * x$ *for some fixed* $a$ *in* $A$, *if and only if* $A$ *is regular.*

*Proof.* Let $A$ be regular and let $\sigma : A \to A$ be an isometry. Choose $a$ in $A$ such that $\sigma(a) = 0$. As $\sigma$ is a bijection such $a$ exists uniquely.
Now for any $x$ in $A$, $\sigma(x) = \sigma(x) * 0 = \sigma(x) * \sigma(a) = x * a$.
Conversely, let every isometry be of that form. The identity $i_A$ is also an isometry and so

$$x = i_A(x) = x * a \text{for some} a.$$

Using $x = 0$ and by $(m_4)$ we get $a = 0$, i.e., $x * 0 = x$ for any $x$ so that $x$ is regular. □

**Remark 4.1.** *Every isometry is a regular autometrized algabra is of the form* $x \mapsto x * a$; *but for every* $a$ *the mapping* $x \mapsto x * a$ *need not be an isometry.*

For instance, in the regular automtrized algebra $(A, +, \leq, 0, *)$ where $A$ is the set of all non negative real numbers, $+, \leq, 0$ have the usual meanings, and $a * b = |a - b|$, the mapping $x \mapsto |1 - x|$ is not an isometry.

**Theorem 4.2.** *If $A$ is a regular autometrized algebra, then, the group of all isometries of $A$ is algebra and also every isometry is an involution.*

*Proof.* We know that from Theorem 2.1, $A^*$ the set of all isometries of $A$ is a group under the composition of mappings.
By Theorem 4.1, every isometry is of the form $\sigma_a$ where $\sigma_a : x \mapsto a * x$. Let $\sigma_a, \sigma_b$ be isometries of $A$, we have $\sigma_a o \sigma_b$ is also an isometry and let it be $\sigma_e$ we have $e * x = a * (b * x)$ for all $x$ in $A$. In particular, $c * 0 = a * (b * 0)$ implying $e = a * b$. As $a * b = b * a$ (by $m_3$), we get $\sigma_a o \sigma_b = \sigma_b = \sigma_a$. (and each equal to $\sigma_{a*b}$ or $\sigma_{b*a}$). Also $(\sigma_a o \sigma_a)(x) = (a * a) * x = 0 * x = x = i_A(x)$ which implies $\sigma_a$ is an involution. Also we have $\sigma^{-1} = \sigma$ for any isometry of $A$. □

**Theorem 4.3.** *Let $A$ be a regular autometrized algebra and let $G = \{a \in A | \text{The mapping} x \mapsto a*x \text{is an isometry}\}$. Then, $(G, *, 0)$ is a group isomorphic with $A^*$ The group of all isometries of $A$.*

*Proof.* Let $a, b, c \in G \implies \sigma_a, \sigma_b, \sigma_c$ are isometries of $A$ where $\sigma_d$ denotes the map $x \mapsto d * x$.
we have $(\sigma_a o \sigma_b) o \sigma_c = \sigma_a o (\sigma_b o \sigma_c)$ as $A^*$ is already a group under the composition of mappings. As $(\sigma_a o \sigma_b) o \sigma_c = \sigma_{(a*b)*c}$ and $\sigma_a o (\sigma_b o \sigma_c) = \sigma_{a*(b*c)}$, follows that $(a * b) * c = a * (b * c)$ for $a, b, c$ in $G$. Also $a \in G \implies a^{-1} (= a)$ belongs to $G$. Hence, $G$ is a group. The correspondence $a \mapsto \sigma_a$ gives the required isomorphism from $G$ in to $A^*$. □

The following theorem is immediate from theorem 2.2.

**Theorem 4.4.** *If $A_1$ and $A_2$ are autometrized algebras and $A_1^*, A_2^*$ are the groups of isometries of $A_1$ and $A_2$ respectively, then, $A_1^* * A_2^*$ is the group of isometries $(A_1 \times A_2)^*$ of $A_1 \times A_2$.*

We say the group of isomtries of an OMM-space is said to be "simply transitive" if given $a, b$ in $A$, there is a unique isometry carrying $a$ to $b$.
Now we have the following:

**Theorem 4.5.** *If $A$ is a regular autometrized algebra and if the group of isometris is simply transitive, then, $(A, *)$ is a commutative loop.*

*Proof.* $a * x = b$ is solvable uniquely and 0 is the identity with respect to $x$, follows the result. □

**Definition 4.3.** *Any autometrized algebra $(A, +, \leq, 0, *)$ is said to be semi-regular if $a \geq 0$ implies $a * 0 = a$. The examples $\mathbb{R}, \mathbb{R} \times \mathbb{R}$, lattice ordered groups, and $DR\ell$-monoids are all semi-regular.*

# 5 Representable autometrized algebra

**Definition 5.1.** *A system $(A, +, \leq, 0, *)$ where*

*(1) $(A, +, \leq, 0)$ is a commutative lattice ordered monoid.*

*(2)* $*$ *is semi-regular and the mappings* $x \mapsto a + x$, $x \mapsto a \cup x$, $x \mapsto a \cap x$, $x \mapsto a * x$ *are all contractions with respect to* $*$.

Note: Here $\cup$ and $\cap$ are the join and meet operations in the lattice $(A, \leq)$. By a lattice ordered monoid, we mean it satisfies the laws

$$a + (b \cup c) = (a + b) \cup (a + c) \text{and}$$

$$a + (b \cap c) = (a + b) \cap (a + c) \text{for } a, b, c \text{in} A$$

Also by saying a mapping $f : A \to A$ is a contraction with respect to an operation $* : A \times A \to A$ we mean $f(x) * f(y) \leq x \vee y$ for all $x, y$ in $A$.
The definition 5.1 implies for a given $a$ and all $x, y$ in $A$, we have

$$(a + x) * (a + y) \leq x * y; (a \cup x) * (a \cup y) \leq x * y; (a \cap x) * (a \cap y) \leq x * y, (a * x) * (a * y) \leq x * y.$$

We can see that Boolean algberas, Brouwerain algberas, Abelian lattice ordered groups, $DR\ell$-monoids and $MV$-algeberas are all representable autometrized algberas.

**Theorem 5.1.** *Let* $A = (A, +, \leq, 0)$ *be a commutative lattice ordered monoid but* $\theta : A \times A \to A$ *be a mapping satisfying the condition* $(R) : (a\theta(a \cap b)) + (a \cap b) = a$ *for all* $a, b$ *in* $A$. *Then* $(A, +, \leq, 0, \theta)$ *is a representable automterized algebra if and only if the mappings* $x \mapsto a + x$, $x \mapsto a \cap x$, $x \mapsto a \cup x$, $x \mapsto a\theta x$ *are all contractions with respect to* $\theta$.

*Proof.* The necessary part follows directly from the Definition 5.1 if $(A, +, \leq, 0, \theta)$ is a representable autometrized algebra. For the sufficiency part, assume that $\theta$ satisfies the condition $(R)$ and all the mappings $x \mapsto a + x$, $x \mapsto a \cap x$, $x \mapsto a \cup x$, and $x \mapsto a\theta x$ are contractions with respect to $\theta$.
We need to show that $\theta$ is semi-regular metric on $A$. Let $a \geq 0$ we have $a = a\theta(a \cap 0) + (b \cap 0) = (a\theta 0) + 0 = ao0$, so that $\theta$ is semi regular (provided $\theta$ is a metric operation).
We now have

$$\begin{aligned} 0 = 0\theta 0 &= \{(0\theta(0 \cap a \cap b)) + (0 \cap a \cap b)\}\theta \\ &\theta(\{(0\theta(0 \cap a \cap b))\} + 0 \cap a \cap b) \\ &\leq (0 \cap a \cap b)\theta(0 \cap a \cap b) \\ &= (0 \cap a \cap b \cap a)\theta(0 \cap a \cap b \cap b) \\ &\leq a\theta b \text{for all} a, b \text{in} A \text{i.e., } \theta \text{ satisfies } a\theta b \geq 0 \text{ for all } a, b \text{ in } A. \end{aligned}$$

Now we show that $a\theta b = 0$ if and only if $a = b$. Clearly, $a\theta a = \{(0\theta(a \cap 0)) + (a \cap 0)\}\theta(a\theta(a \cap 0) + (a \cap 0)) \leq (a \cap 0)\theta(a \cap 0) \leq 0\theta 0 = 0$. Thus $a\theta a = 0$ for all $a$.
To prove that $a\theta b = 0$ implies $a = b$, we first prove that $a\theta(a \cap b) = (a \cap b)\theta b$ and $a\theta b = b\theta a$ for all $a, b$ in $A$.
Now $a\theta(a \cap b) = (a \cap (a \cup b))\theta(a \cap b) \leq (a \cup b)\theta b = (b \cup a)\theta(b \cup (a \cap b)) \leq a\theta(a \cap b)$. Hence, $a\theta(a \cup b) = (a \cup b)\theta b$ for all $a, b$ in $A$.
Now $a\theta b = \{a\theta(a \cap b) + (a \cap b)\}\theta\{b\theta(a \cap b) + (a \cap b) \leq \{a\theta(a \cap b)\}\theta\{b\theta(a \cap b)\} \leq ((a \cup b)\theta b) + ((a \cup b)\theta a) \leq b\theta a$

for all $a, b$ in $A$ and hence by interchanging $a$ and $b$, we get $a\theta b = b\theta a$ for all $a, b$ in $A$.
Now let

$$\begin{aligned} a\theta b = 0 &\implies a = (a\theta(a \cap b)) + (a \cap b) \leq (a\theta b) + b = 0 + b \\ &\implies a \leq b. \end{aligned}$$

Again by interchanging $a$ and $b$, we get $b \leq a$ so that $a = b$.
For the triangle inequality, let $a, b, c \in A$, we have

$$\begin{aligned} a\theta b &= (a\theta b)\theta((a\theta b) \cap (c\theta b)) + (a\theta b) \cap (c\theta b) \\ &\leq (a\theta b) \cap (c\theta b) + (c\theta b) \\ &= (b\theta a) \cap (b\theta c) + (c\theta b) \\ &\leq (a\theta c) + (c\theta b) \end{aligned}$$

Thus $\theta$ satisfies $(m_1)$ through $(m_4)$. Hence, the theorem. □

**Theorem 5.2.** *Let $(A, +, \leq, 0, *)$ be a separable autometrized algebra satisfying*

$$(R) : a * (a \cap b) + (a \cap b) = a \text{for alla}, b \text{in} A.$$

*Let $\theta : A \times A \to A$ be a mapping satisfying*

$$(R') : a\theta(a \cap b) + (a \cap b) = a \text{for alla}, b \text{in} A$$

*and such that the mappings $x \mapsto a + x$, $x \mapsto a \cap x$, $x \mapsto a \cup x$, and $x \mapsto a\theta x$ for any $a$ are all contractions with respect to $\theta$. Then $\theta = *$.*

We need a few lemmas for the proof of Theorem 5.2.

**Lemma 5.1.** *$a * (a \cap b) = (a \cup b) * b$ for all $a, b$ in $A$.*

*Proof.* $a * (a \cap b) = (a \cap (a \cup b)) * (a \cap b) \leq (a \cup b) * b = (b \cup a) * (b \cup (a \cap b)) \leq a * (a \cap b)$.
$\implies a * (a \cap b) = (a \cup b) * b$ □

**Lemma 5.2.** *If $b \leq a$, then $a = (a * b) * b$*

*Proof.* It is obvious. □

**Lemma 5.3.** *$a \cup b = (a * b) + (a \cap b)$ for all $a, b$ in $A$.*

*Proof.* Since $b \leq a \cup b$, $a \leq a \cup b$ we have $a \cup b = \{(a \cup b) * b\} + b = (a * b) * b$, and $a \cup b = ((a \cup b) * a) + a \leq (a * b) + a$.
Hence, $a \cup b \leq ((a * b) + b) \cap ((a * b) + a) = (a * b) + (a \cap b)$. □

**Lemma 5.4.** *$a * b = (a \cup b) * (a \cap b)$ for all $a, b$ in $A$.*

*Proof.*

$$\begin{aligned} a*b &= \{(a\times(a\cap b))+(a\cap b)\}*\{(b\times(a\cap b))+(b\cap a)\} \\ &\leq (a*(a\cap b))*(b*(a\cap b)) \\ &= ((a\cup b)*b)\times((a\cap b)*b) \\ &\leq (a\cup b)*(a\cap b) \\ &= ((a*b)+(a\cap b))*(a\cap b) \\ &\leq (a*b)*0 = a*b(\text{by semi regularity of}*) \end{aligned}$$

Hence, $(a\cup b)*(a\cap b) = a*b$. □

**Lemma 5.5.** *$a*b = (a*(a\cap b))+((a\cap b)*b)$ for all $a,b$ in $A$.*

*Proof.* For $a,b$ in $A$,$a*b \leq (a*(a\cap b))+((a\cap b)*b)$. Further,

$$\begin{aligned} (a*(a\cap b))+((a\cap b)*b) &= ((a\cup b)*b)+((a\cap b)*b) \\ &= ((a*b)+(a\cap b))*(b*(a\cap b)+(a\cap b))+b*(a\cap b) \\ &\leq (a*b)*(b*(a\cap b))+(b*(a\cap b)) \\ &= a*b\text{since}b*(a\cap b)\leq b*a = a*b \end{aligned}$$

Hence, $a*b = (a*(a\cap b))+((a\cap b)*b)$. □

Proof of Theorem 5.2 Now $(A,+,\leq,0,\theta)$ is a separable autometrized algebra satisfying $(a\theta(a\cap b))+(a\cap b)=a$ for all $a,b$ in $A$. Hence, the above lemmas are valid in the autometrized algebra $(A,+,\leq,0,\theta)$ (with $\theta$ in place of $*$). So, $a*0=a=a\theta 0$ for $a\geq 0$, and for any $a,b$ in $A$.

$$\begin{aligned} a\theta b &= (a\cup b)\theta(a\cap b) = ((a*b)+(a\cap b))\theta(a\cap b) \\ &\leq (a*b)\theta 0 = a*b = (a\cup b)*(a\cap b) \\ &= ((a\theta b)+(a\cap b))*(a\cap b) \\ &\leq (a\theta b)*0 = a\theta b \end{aligned}$$

Hence, $a\theta b = a*b$. The proof is complete.
Combining the Theorems 5.1 and 5.2 we can formulate the following

**Theorem 5.3.** *Let $(A,+,\leq,0)$ be a commutative lattice ordered monoid. Then, there exists at the most one operation $*$: $A\times A\to A$ such that $(A,+,\leq,0,*)$ is a separable autometrized algebra satisfying the condition $(a*(a\cap b))+(a\cap b)=a$.*

# Chapter 2

# Boolean Algebras, Semi Browerian Algebras and Browerian Algebras

## 1 Boolean Algebras

We assume basic knowledge of Boolean Algebras. A Boolean Algebra is a distributive and complemented lattice. There are several axiom systems for Boolean algebras, some are very lengthy but convenient to use and some are very brief but difficult to use. We give below a popular and convenient system of axioms.
An algebra $(B, \cup, \cap, ', 0, 1)$ of spaces $(2, 2, 1, 0, 0)$ is called a Boolean Algebra iff

(i) $a \cup a = a$; $a \cap a = a$

(ii) $a \cup b = b \cup a$; $a \cap b = b \cap a$

(iii) $a \cup (b \cup c) = (a \cup b) \cup c$; $a \cap (b \cap c) = (a \cap b) \cap c$

(iv) $a \cup (a \cap b) = a$; $a \cap (a \cup b) = a$

(v) $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$

(vi) $a \cup a' = 1$, $a \cap a' = 0$

These axioms will imply, $(a')' = a$, complementation is unique and Demorgan's Laws: $(a \cap b)' = a' \cup b'$, $(a \cup b)' = a' \cap b'$ and obtain distributive law $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$, and $0 \leq a \leq 1$; $a \leq b \iff a \cap b' = 0 \iff a \cap b = a, a \cup b = b$.

**Theorem 1.1.** *Let $B = (B, \cup, \cap, ', 0, 1)$ be a Boolean Algebra. For $a, b$ in $B$, write $a * b = (a \cap b') \cup (a' \cap b)$. $a * b$ is called the symmetric difference of $a$ and $b$. We can observe:*

*($m_1$)* $a * a = 0$

*($m_2$)* $a * c \leq (a * b) \cup (b * c)$

*($m_3$)* $a * b = b * a$

$(m_4)$ $a * b = 0 \implies a = b$

*These shows that* $(B, \cup, \cap, ', 0, 1)$ *is an autometrized algebra* $(B, \cup, 0, \leq, *)$ *and also a metric OMM-space* $(B, B, *)$.

Though the proof of the above observation is straight forward, for the sake of completeness and reference, we establish theorem and include the proof also.

**Theorem 1.2.** *Any Boolean Algebra is a regular autometrized algebra with the symmetric difference as distance function.*

*Proof.* As any $x$ in $B$ (a Boolean Algebra) satisfies $0 \leq x \leq 1$, we have $a*b \geq 0$. Also $a*a = (a \cap a') \cup (a' \cap a) = 0$.
$a * b = b * a$ is also clear.
For proving $(m_2)$ the triangle inequality, it is convenient to use the notation $a - b = a \cap b'$.
With the notation, $a * b = (a - b) \cup (b - a)$ we first observe that given $a, b$, there is a least $x$ such that $x \cup b \geq a$ and this $x$ is $a - b (= a \cap b')$. $(a - b) \cup b = (a \cap b') \cup b = (a \cup b) \cap (b \cup b') = (a \cup b) \cap 1 = a \cup b \geq a$.
Now let $x \cup b \geq a \implies (x \cup b) \cap b' \geq a \cap b' \implies a \cap b' \leq (b' \cap x) \cup (b' \cap b) = b' \cap x \leq x$. Thus $a \cap b'$ is the least $x$ such that $x \cup b \geq a$.
Now we observe that for any $a, b, c$ in $B$
$a - c \leq (a - b) \cup (b - c)$.
Now $c \cup (a - b) \cup (b - c) = c \cup (b - c) \cup (a - b) \geq b \cup (a - b) \geq a$
$\implies a - c \leq (a - b) \cup (b - c)$.
Now $(a * b) \cup (b * c) = (a - b) \cup (b - a) \cup (b - c) \cup (c - b) \geq (a - c) \cup (c - a) = a * c$. Thus $m_2$ and all other $m_1$, $m_3$, $m_4$ are satisfied. For $(m_4)$, $a * b = 0 \implies a \cap b' = 0 = a' \cap b \implies a \leq b$ and $a = b$. Also $a * 0 = (a \cap 0') \cup (a' \cap 0) = (a \cap 1) \cup 0 = a \cup 0 = a$ so that $*$ is regular. □

**Theorem 1.3.** *For each* $a$ *in* $B$*(a Boolean Algebra). The mapping* $x \mapsto a*x$ *is an isometry and these mappings are all the isometries of* $B$.

*Proof.* By Theorem 1.5.4, every isometry is of the form $\sigma_a : x \mapsto a * x$. Now, we show that for each $a$, the mapping $\sigma_a$ is an isometry.
For this, we need to show that $*$ is associative. Let $a, b, c \in B$. We have $(a*b)*c = [(a*b) \cap c'] \cup [(a*b)' \cap c] = [((a \cap b') \cup (a' \cap b)) \cap c'] \cup [(a' \cup b) \cap (a \cup b') \cap c] = (a \cap b' \cap c') \cup ((a' \cap b) \cap c') \cup (a' \cap b' \cap c) \cup (c \cap b \cap a)$. Now $a * (b * c) = (b * c) * a = (c * b) * a = (c \cap b' \cap a') \cup ((c' \cap b) \cap a') \cup (c' \cap b' \cap a) \cup (c \cap b \cap a) = (a * b) * c$ in view of commutativity and associativity of $\cup$, $\cap$. Hence, $(B, *, 0)$ is an abelian group with inverse of $a$ is equal to $a$.
For for $a, x, y$ in $B$
$\sigma_a(x) * \sigma_a(y) = (a * x) * (a * y) = a * (a * x) * (a * y) = x * a * a * y = x * 0 * y = x * y$ so that $\sigma_a$ is distance preserving. $\sigma_a$ is a surjection as for any $z$ in $B$ $\sigma_a(a * z) = z$, so that $\sigma_a$ is an isometry. □

We now immediately have the following

**Theorem 1.4.** *The group of isometries of a Boolean Algebra* $B = (B, \cup, \cap, ', 0, 1)$ *is isomorphic with the group* $(B, *, 0)$.

It is well known that given any Boolean Algebra $(B, \cup, \cap, ', 0, 1)$, the algebra $(B, *, \cap, 0, 1)$ is a ring with unity (called the corresponding Boolean Ring) in which every element is idempotent. Thus it is clear that the group of isometries is isomorphic with the additive group of the corresponding Boolean ring.

# 2 Betweenness

As in OMM-spaces, in a Boolean algebra we have the concept of (metric) betweenness. As any Boolean algebra is also a distributive lattice, we have the concept of lattice betweenness. Accordingly, for any $a, b, c$ in $B$, $b$ lies between (lattice) $a$ and $c$ (in symbols $(a, b, c)L$) if and only if $a \cap c \leq b \leq a \cup c$. In the case of Boolean algebras we show both the concepts lattice betweenness and metric betweenness coincide.Before going to that it is convenient to have the following lemma. Note that we have introduced a "-" in a Boolean algebra where $a - b = a \cap b'$.

**Lemma 2.1.** *(i)* $(a - b) \cup b = a \cup b$

*(ii)* $a \leq x \leq b \implies b - a = (b - x) \cup (x - a)$

*(iii)* $a - (b \cup c) = (a - b) - c$

*(iv)* $a - (a - y) = a \cap y$

*(v)* $(a \cup b) - c = (a - c) \cup (b - c)$

*(vi)* $a - (b \cap c) = (a - b) \cup (a - c)$

Proof is routine and straight forward using DeMorgan's laws and $a - b = a \cap b'$. For instance $(iii)$: $a - (b \cup c) = a \cap (b \cup c)' = a \cap b' \cap c' = (a \cap b') - c = (a - b) - c$.
For $(iv)$, $a - (a - y) = a - (a \cap y') = a \cap (a \cup y')' = a \cap (a' \cup y) = (a \cap a') \cup (a \cap y) = a \cap y$.
For $(ii)$, let $a \leq x \leq b$. Now

$$\begin{aligned}(b - x) \cup (x - a) &= (b \cap x') \cup (x \cap a') \\ &= (b \cup x) \cap (x \cup x') \cap (b \cup a') \cap (x' \cup a') \\ &= b \cap [((b \cup a') \cap a') \cup ((b \cup a') \cap x')] \\ &= b \cap (a' \cup b) \cap (a' \cup x') = b \cap (a' \cup x') = b \cap a' \text{ as } a \leq x \implies x' \leq a' \\ &= b - a\end{aligned}$$

We omit the proof for the rest.

**Theorem 2.1.** *For $a, x, b$ in $B$, $(a, x, b)L \iff (a, x, b)M$.*

*Proof.* Assume $(a, x, b)L$ holds
$\implies a \cap b \leq x \leq a \cup b$
$\implies [(a \cup b) - x] \cup [x - (a \cap b)] = (a \cup b) - (a \cap b)$

$\Longrightarrow (a-x)\cup(b-x)\cup(x-a)\cup(x-b)=(a\cup b)-(a\cap b)$

$\Longrightarrow (a-x)\cup(x-a)\cup(b-x)\cup(x-b)=(a\cup b)-(a\cap b)$

$\Longrightarrow (a*x)\cup(x*b)=a*b$

$\Longrightarrow (a,x,b)M$ Now assume $(a,x,b)M$

$\Longrightarrow (a*x)\cup(x*b)=a*b$

$\Longrightarrow (a-x)\cup(x-a)\cup(x-b)\cup(b-x)=(a\cup b)-(a\cap b)$

$\Longrightarrow ((a\cup b)-x)\cup(x-(a\cap b))=(a\cup b)-(a\cap b)$

$\Longrightarrow ((a\cup b)-x)\cup(x-(a-b))\cup(a\cap b)=a\cup b$ (as$[(a\cup b)-(a\cap b)]\cup(a\cap b)=a\cup b$)

$\Longrightarrow a\cup b\geq x-(a\cap b)$

$\Longrightarrow a\cup b\geq(x-(a\cap b))\cup(a\cap b)=x\cup(a\cap b)$

$\Longrightarrow a\cup b\geq x$

Also $(a\cup b)-(a\cap b)\geq(a\cup b)-x$

$\Longrightarrow ((a\cup b)-x)-[(a\cup b)-(a\cap b)]\leq 0$

$\Longrightarrow (d-x)-(d-y)\leq 0$ with $d=a\cup b$, $y=a\cap b$

$\Longrightarrow (d-(d-y))-x\leq 0$ (lemma)

$\Longrightarrow (d\cap y)-x\leq 0$

$d\cap y\leq x$

$a\cap b\leq x$

$(a,x,b)L$ □

**Definition 2.1.** *For any three distinct elements $a,b,c$ of a Boolean Algebra $B$, the triangle $\delta(a,b,c)$ is said to have finity if each side is equal to opposite vertex, i.e., if $a*b=c$, $b*c=a$, $c*a=b$. The triangle $\delta(a*b,b*c,c*a)$ is called the first distance triangle of $\delta(a,b,c)$ and so on.*

**Theorem 2.2.** *In $B$, every first distance triangle has finty.*

*Proof.* Obvious as $*$ is a group operation. □

# 3 Boolean Rings

A ring in which every element is idempotent is called a Boolean Ring.

Every Boolean Ring need not have unity. It is well known that every Boolean ring is of characteristic two (i.e., $2a=0$ for all $a$) and commutative. Also it is well known that if $(B,,+,',0,1)$ is a Boolean ring with unity. Then $(B,\cup,\cap,',0,1)$ is a Boolean algebra where $a\cup b=a+b+ab$, $a\cap b=a\ldots b$, $a'=1+a$. Also if $(B,\cup,\cap,',0,1)$ is Boolean algebra, then $(B,+,\ldots,0,1)$ is Boolean algebra with unity where $a+b=(a\cap b')\cup(a'\cap b')$, $a\ldots b=a\cap b$. Also the processes are inverses of each other, i.e., if one starts from a Boolean Algebra $*$ is the original Boolean algebra ... and vice versa.

**Theorem 3.1.** *Let $(B,+,\ldots,0)$ be a Boolean ring. If we write $d(a,b)=a+b$, then $(B,\cup,0,d)$ is autometrized algebra, where $a\cup b=a+b+ab$.*

*Proof.* $\leq$ is defined by $a \leq b \iff a \cup b = b$ ( or equivalently $a \ldots b = a$). $\leq$ is a lattice order with $a \ldots b = a \cap b$. Now $d(a,a) = a + a = 0$. Clearly $d(a,b) = d(b,a)$. A direct calculation using $x^2 = x$ and $x + x = 0$ yields for any $a, b, c$,

$$(a + c) \ldots [(a + b) \cup (b + c)] = a + c$$

implying $d(a,c) \leq d(a,b) \cup d(b,c)$. □

We get the following theorem immediately.

**Theorem 3.2.** $(B, \cup, 0, d)$ *is regular.*

*Proof.* $d(a,0) = a + 0 = a$. □

**Theorem 3.3.** *Every isometry* $\sigma$ *of* $(B, \cup, 0, d)$ *is given by* $\sigma(x) = d(0,x)$ *(i.e.,* $\sigma(x) = a + x$*) for some* $a$.

**Theorem 3.4.**

*(i) Every translation* $x \mapsto x + a$ *is an isometry.*

*(ii) The group of isometries of* $(B, \cup, 0, d)$ *is isomorphic with the additive group* $(B, +, 0)$ *of the Boolean ring* $(B, +, \ldots, 0)$.

# 4 Browerian and semi Browerian algebras

**Definition 4.1** (semi Brouwerian Algebra)**.** *An algebra* $(L, +, 0, -)$ *of spaces* $(0, 0, 2)$ *where* $(L, +, 0)$ *is a join semilattice with* $0$ *as identity and* $a - b$ *for any given* $a, b$ *in* $L$ *is the least element* $x$ *in* $L$ *such that* $x + b \geq a$.

Note: Here $\leq$ is the order induced by the join operation $+$. From the definition we have the following consequences.

**Theorem 4.1.** *(1)* $a - a = 0$

*Proof.* $a + 0 = a \implies a - a \leq 0$. As $(L, +, 0)$ is a join semilattice, 0 least element so that $0 \leq a - a \implies a - a = 0$. □

*(2)* $(a - b) + b = a + b$

*Proof.* $a + b$ is the supremum of $a$ and $b$. Also $a \leq b \implies x + a \leq x + b$. Now $(a - b) + b \geq a$. $a - b \geq 0 \implies (a - b) + b \geq 0 + b = b$ so that $(a - b) + b = a + b$. □

*(3)* $(a - c) - b = a - (b + c)$

*Proof.* $b + c + [(a - c) - b] = c + [(c - a) - b] + b \geq c + (a - c) \geq a \implies a - (b + c) \leq (a - c) - b$. Now $[a - (b + c)] + b + c \geq a \implies a - c \leq [a - (b + c)] + b$ and so $(a - c) - b \leq a - (b + c)$. Hence, $a - (b + c) = (a - c) - b$ □

*(4)* $a - b = 0 \implies a \leq b$

*Proof.* $a \leq b \implies b + 0 \geq a \implies a - b \leq 0 \implies a - b = 0$. Also $a - b = 0 \implies (a - b) + b = 0 + b = b \implies a + b = b \implies a \leq b$. □

*(5)* $(a + b) - c = (a - c) + (b - c)$

*Proof.* $c + (a - c) + (b - c) = c + (a - c) + c + (b - c) \geq a + b \implies (a + b) - c \leq (a - c) + (b - c)$. Also $a - c \leq (a + b) - c$, $b - c \leq (a + b) - c \implies (a - c) + (b - c) \leq (a + b) - c$. Hence the result. □

# 5 Browerian Algebras

Semi- Brouwerian algebras being the generalization of Brouwerian Algebras, they both share many properties in common. Brouwerian Algebras have more algebraic structure than semi-Brouwerian Algebras, naturally in Brouwerian Algebras we get more results.

**Theorem 5.1.** *A lattice $(L, \vee, \wedge, 0, 1)$ with least and greatest elements 0 and 1 respectively is called Brouwerian Algebra if and only if given $a, b$ in L, there is a least element $x$ (denoted by $a - b$) such that $x \vee b \geq a$.*

***Note.*** *If $(L, \vee, \wedge, -, 0, 1)$ is a Brouwerian Algebra, then the reduct $(L, \vee, -, 0)$ is a semi-Brouwerian Algebra. A semi-Brouwerian Algebra becomes a Brouwerian Algebra if under the same inclusion $\leq$, is a lattice and has a greatest element.*

*Note that all the results on semi-Brouwerian Algebras (1) through (8) are valid in Brouwerian Algebras also. Now we have some additional consequences of Brouwerian Algebras.*

**Theorem 5.2.** *Let $(L, \vee, \wedge, -, 0, 1)$ be a Brouwerian Algebras and $a, b, c, ...$ be elements of L.*

**Theorem 5.3.** $a - (b \wedge c) = (a - b) \vee (a - c)$

*Proof.* $b \wedge e \leq b,\ e \Rightarrow a - (b \wedge e) \geq a - b, a - e$ (by definition of $-$). This implies $a - (b \wedge e)$ is an upper bound of $a - b$ and $a - e$.

In fact $b \vee (a - (b \wedge e)) \geq (b \wedge e) \vee (a - (b \wedge e)) \geq a$. This implies $a - b \leq a - (b \wedge e)$.

Similarly, $a - e \leq a - (b \wedge e)$.

For the other inequality, we need $(L, \vee, \wedge)$ is a distributive lattice which we prove in the next theorem.

Now consider $(b \wedge e) \vee (a - b) \vee (a - e) = (b \wedge e) \vee d$, where $d = (a - b) \vee (a - e)$. This implies $(b \wedge e) \vee (a - b) \vee (a - e) = (d \vee b) \wedge (d \vee e) = ((a - b) \vee b \vee (a - e)) \wedge ((a - b) \vee (a - e) \vee e) \geq ((a - b) \vee b) \wedge ((a - e) \vee e) \geq a \wedge a = a$. This implies $a - (b \wedge e) \leq (a - b) \vee (a - e)$. Hence the result. □

**Theorem 5.4.** *$(L, \vee, \wedge)$ is a distributive lattice.*

*Proof.* We always have $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee e)$ for any $a, b, c$ in L. It is enough to show that $a \vee (b \wedge e) \geq (a \vee b) \wedge (a \vee e)$. Put $d = (a \vee b) \wedge (a \vee c) \Rightarrow a \vee b \geq d,\ a \vee e \geq d \Rightarrow d - a \leq b,\ d - a \leq c \Rightarrow d - a \leq b \wedge e \Rightarrow a \vee (b \wedge e) \geq d = (a \vee b) \wedge (a \vee c)$. □

**Theorem 5.5.** *We write $\neg a = 1 - a$ for any $a$ in $L$.*

*i)* $a \leq b \Rightarrow \neg b \leq \neg a$.

*ii)* $a \vee \neg a = 1, \neg 0 = 1, \neg 1 = 0$.

*iii)* $\neg\neg a \leq a$.

*iv)* $\neg\neg\neg a = \neg a$.

*v)* $\neg(a \wedge b) = \neg a \vee \neg b,\ \neg(a \wedge \neg a) = 1$.

*vi)* $\neg(a \vee b) = \neg\neg(\neg a \wedge \neg b)$.

*Proof.* We prove (vi) as the other are straight from the definition.
$\neg\neg(\neg a \wedge \neg b) = \neg(\neg\neg a \vee \neg\neg b) = 1 - (\neg\neg a \vee \neg\neg b) = (1 - \neg\neg a) - \neg\neg b = \neg\neg\neg a - \neg\neg b = \neg a - \neg\neg b = (1 - a) - \neg\neg b = (1 - \neg\neg b) - a = \neg\neg\neg b - a = \neg b - a = (1 - b) - a = 1 - (a \vee b) = \neg(a \vee b)$. □

# 6 Metrization

For any $a, b$ in a semi- Brouwerian algebra $(S, \vee, -, 0)$ (or in a Brouwerian algebra, $(B, \vee, \wedge, -, 0, 1)$), write $a \star b = (a - b) \vee (b - a)$. $a \star b$ is called the symmetric difference of $a$ and $b$.

**Theorem 6.1.** *Any semi-Brouwerian algebra (or a Brouwerian algebra) is a Qutometrized algebra with the symmeric difference as a distance function.*

*Proof.* $(m_1)$ : For any $a$, $a \star a = (a - a) \vee (a - a) = 0$.

$(m_2)$ : For any $a, b, c$, we have $a - c \leq (a - b) \vee (a - e)$ as $e \vee (a - b) \vee (b - e) = (a - b) \vee e \vee (b - e) \geq (a - b) \vee b \geq a$. This implies $a - c \leq (a - b) \vee (b - e)$. Similarly, $c - a \leq (c - b) \vee (b - a)$. So, $a \star c = (a - c) \vee (c - a) \leq (a - b) \vee (b - e) \vee (e - b) \vee (b - a) = (a \star b) \vee (b \star c)$.

$(m_3)$ : $a \star b = b \star a$.

$(m_4)$ : $a \star b = 0 \Rightarrow (a - b) \vee (b - a) = 0 \Rightarrow a - b = 0, b - a = 0 \Rightarrow a \leq b$ and $b \leq a \Rightarrow a = b$.

□

**Theorem 6.2.** *$\star$ is regular.*

*Proof.* $a \star 0 = (a - 0) \vee (0 - a) = 0 \vee 0 = 0$. □

**Theorem 6.3.** *Every isometry of $S$ and $B$ is given by $x \mapsto a \star x$ for some $a$ (in $S$ or $B$ as the case may be).*

*Proof.* By Theorem 1.5.4.
Note that there exists an element $a$ for which $x \mapsto a \star x$ is not an isometry. For instance in Brouwerian chain, with at least three elements, for any a, we have $1 - a = 1$ (for $0 \leq a < 1$). i.e. The mapping $x \mapsto 1 - x (= 1 \star x)$ is not an isometry. □

Recall (definition 1.5.8) that the group of isometries is simply transitive if given $a, b$ : there is a unique isometry carrying a to b: we have such that the group of isometries of a Boolean algebra is simply transitive.
We now have

**Theorem 6.4.** *A Brouwerian algebra $(L, \vee, \wedge, -, 0, 1)$ is a Brouwerian algebra $(L, \vee, \wedge, \neg, 0, 1)$ if and only if the group of isometries is simply transitive.*

*Proof.* Assume that the group of isometries of Brouwerian algebra $(L, \vee, \wedge, -, 0, 1)$ is simply transitive. It is enough if we show that every element in L by complemented (as already it is shown that L is a distributive lattice).
As $x \vee \neg x = 1$, we show that $x \wedge \neg x = 0$. Put $\sigma$ be an isometry with $\sigma(1) = 0$. This implies $\sigma(\sigma(1)) = 1$ (as $\sigma$ is an involution by Theorem 1.5.5). This implies $\sigma(0) = 1$.
For any $x$, $\sigma(x) \star 0 = \sigma(x) \star \sigma(1) = x \star 1$. This implies $\sigma(x \wedge \neg x) = (x \wedge \neg x) \star 1 = 1 - (x \wedge \neg x) = \neg x \vee \neg\neg x = 1 = \sigma(0) \Rightarrow x \wedge \neg x = 0$ as $\sigma$ is an injection.
The converse part is obvious because in a Boolean algebra the group of isometries is simply transitive. □

**Theorem 6.5.** *A Brouwerian algebra is a Boolean algebra if and only if the mapping $x \mapsto 1 - x$ is an isometry.*

*Proof.* If $x \mapsto 1 - x$ is an isometry, then for any a , $\neg(a \wedge \neg a) = \neg a \vee \neg\neg a = 1 = \neg 0 \Rightarrow \neg a \wedge \neg\neg a = 0$ so that a is complemented and hence B is a Boolean algebra.
The other part is clear as in Brouwerian algebra every mapping $x \mapsto a \star x$ (in particular $1 \star x$) is an isometry.
As Brouwerian contains a lattice structure and also a metric, we can obtain lattice betweenness and metric betweenness. As any Brouwerian algebra is also a distributive lattice, lattice betweenness is (a, b, e)L if and only if $a \wedge e \leq b \leq a \vee e$ in a Boolean algebra, we have shown that lattice betweenness and metric betweenness coincide.

□

**Theorem 6.6.** *A Brouwerian algebra is a Boolean algebra if and only if lattice betweenness coincide with metric betweenness. Assume that in B (Brouwerian algebra) both the notions coincide. For any a in B, we have $(a \wedge \neg a, 0, 1)M$ holds so $(a \wedge \neg a, 0, 1)L$ holds so that $a \wedge \neg a \leq 0$. i.e. $a \wedge \neg a = 0$ so a is complemented and hence B is a Boolean algebra. The other part is a true Boolean algebra results.*

**Theorem 6.7.** *A Brouwerian algebra is a chain if and only if all triangles are isosceles.*

*Proof.* Assume that every triangle is an isosceles in a Brouwerian algebra B. For nonzero elements $a \neq b$, the $\triangle(0, a, b)$ is isosceles by hypothesis. Hence $a \star b = a$ or $b$. If $a \star b = a$, then $b < a$. If $a \star b = b$, then $b < a$. So, B is a chain.
Conversely, if B is a chain, let $a, b, e$ be distinct elements of B. We may assume that $a < b < e$ (as B is a chain). Then $b \star c = e - b = c$, $a \star e = c - a = e$ so that $\triangle(a, b, c)$ is an isosceles. □

We also use the term Brouwerian geometry for Brouwerian algebra metrized with symmetric difference, Boolean geometry for Boolean algebra metrized with symmetric difference. Now we have:

**Theorem 6.8.** *A Brouwerian geometry does not contain equilateral triangles.*
*Let $a, b, c$ be pairwise distinct elements and $x = a \star b = b \star c = c \star a$. Now $x = b \star e = (b - e) \vee (e - b) = (x - c) \vee (x - b) = (a - e) \vee (a - b) \leq a$.*
*Similarly, $x \leq b, x \leq c$. Now $x \vee (a \wedge b) = (a \star b) \vee (a \wedge b) = ((a \vee b) - (a \wedge b)) \vee (a \wedge b) = a \vee b$. This implies $a \wedge b = a \vee b \Rightarrow a = b$, a contradiction.*

**Theorem 6.9.** *A Brouwerian geometry is a Boolean geometry if and only if it is free of isosceles triangles.*

*Proof.* A Boolean geometry does not contain isosceles triangles as $\star$ is a group operation.
Conversely, assume that a Brouwerian geometry B does not contain isosceles triangles. Note that for any $x$, $1 = 1 \star 0 = 1 \star (x \wedge \neg x) = \neg(x \wedge \neg x) = \neg x \vee \neg\neg x = 1$. This implies $x \wedge \neg x$ must be zero. otherwise 1,0, $x \wedge \neg x$ for an isosceles triangle which is a contradiction with the hypothesis. $x \wedge \neg x = 0$ implies $x$ is complemented and so B is a Boolean geometry. □

**Theorem 6.10.** *A Brouwerian geometry B is a Boolean geometry if and only if every first distance triangle has fixity.*

*Proof.* On a Boolean geometry, every first distance triangle has fixity ($\star$ is associative and regular).
Conversely, assume that in B every first distance triangle has fixity. If it is not a Boolean geometry, it would contain an isosceles triangle with sides $x, x, y$ implying by hypothesis $x \star x = y = 0$ and the triangle degenerates. Hence B is a Boolean geometry. □

**Theorem 6.11.** *A Brouwerian algebra is a Boolean algebra if and only if it admits a metric group operation.*

*Proof.* Let B be a Brouwerian algebra and $\psi : B \times B \to B$ be a metric group operation. We have $a\psi 0 = a$(as 0 is identity for $\psi$). This implies $\psi$ is a regular metric. Also, $a\psi a = 0$ as $\psi$ is a metric operation.
By $(m_2)$, force, we have, $(a\psi b) \vee (b\psi 0) \geq a\psi 0$. i.e. $(a\psi b) \vee b \geq a \Rightarrow a - b \leq a\psi b$. Similarly,$b - a \leq a\psi b$ (as $a\psi b = b\psi a$). This implies $(a - b) \vee (a - b) \leq a\psi b$. That is, $a \star b \leq a\psi b$. Now $1 = 1\psi 0$ and $1\psi(a \wedge \neg a) \geq 1 \star (a \wedge \neg a) = 1$. This implies $1\psi(a \wedge \neg a) = 1$ so that $0 = a \wedge \neg a$. As $a \vee \neg a = 1$ follows that $a$ is complemented. Hence B is a Boolean algebra.
For the converse part, in any Boolean algebra $\star$ is a metric group operation. □

**Remark 6.1.** *G. Eliolt has shown that, in every Boolean algebra, symmetric difference is the only metric group operation. So, in the above theorem, it follows that $\psi$ is the symmetric difference also.*

Also in this emusion Theorem 1.6.2(iii) is a result similar to that of Eliolt and it is applicable to Brouwerian algebras also.

**Theorem 6.12.** *In a semi- Brouwerian algebra $(L, \vee, -, 0)$ if $d : L \times L \to L$ is a mapping such that*

*i)* $d(a, a) = 0,\ d(a, b) = d(b, a),\ d(a, 0) = a$

*ii)* $d(a, b) \vee a = d((a \vee c), b) \vee c$, *then* $d(a, b) = (a - b) \vee (b - a)$ *(symmetric difference).*

*Proof.* We have $d(a,b) \vee b = d(a \vee b, b) \vee b = d(a-b, 0) \vee b = (a-b) \vee b = a \vee b \geq a$. This implies $a - b \leq d(a,b)$. Similarly, $b - a \leq d(a,b)$. Hence $(a-b) \vee (b-a) \leq d(a,b)$. Now $d(a,b) = d(a,b) \vee (a \star b) = d(a,b) \vee (a-b) \vee (b-a) = d(a \vee (b-a), b) \vee (b-a) \vee (a-b) = d(a \vee b, b \vee (a-b)) \vee (a \star b) = d(a \vee b, a \vee b) \vee (a \star b) = a \star b$. □

**Theorem 6.13.** Notation: *In a semi- Brouwerian algebra $L = (L, +, -, 0)$ write $a \cdot b = (a+b) - (a \star b)$, where $a \star b$ is the symmetric difference $(a-b) + (b-a)$.*

**Lemma 6.1.** *$a \cdot b = a - (a \star b) = b - (a \star b)$.*

*Proof.* Observe that $a \cdot b = b \cdot a$.
We have $a - (a \star b) = (a + (a \star b)) - (a \star b) = (a + (a-b) + (b-a)) - (a \star b) = (a+b) - (a \star b)$. Likewise $b - (a \star b) = a \cdot b$. □

**Theorem 6.14.** *If $a - (b-a) = a$ for all $a, b$ in $L$, then $L$ is a Boolean ring.*

We need three more Lemmas to prove the Theorem.

**Lemma 6.2.** *$a \cdot (bc) = (ab) \cdot c$ for all $a, b, c$ in $L$.*

*Proof.* By hypothesis $a - (a-b) = (a - (b-a)) - (a-b) = a - ((a-b) + (b-a)) = a - (a \star b)$. By Lemma (6.1) above, $ab = a - (a-b) = b - (b-a)$ (by symmetry).
Now $(ab)c = ab - ((ab) - c) = [b - (b-a)] - ([b - (b-a)] - c)[b - (b-a)] - \{b - [c + (b-a)]\} = [b - (b-a)] - \{b - [(b-c) - c]\} = [a - (a-b)] - [(b-c) - (b-a)] = b - [(b-c) - ((b-a) + (b-c))] = b - [(b-c) + (b-a)] = b - [(b-a) + (b-c)] = (cb)a = a(bc)$. □

**Theorem 6.15.** *$ab$ is the greatest lower bound of $a$ and $b$.*

*Proof.* First, if $ab = a$, then $b - (a \star b) = a$, so that $a \leq b$ and $a - (b-a) = a$. Then $ab = a - (a-b) = a - 0 = a$. Obviously $ab$ is a lower bound of a and b, and now let $t$ be a lower bound of a and b. Then $ta = t$ and $tb = t$ so that $t(ab) = (ta)b = tb = t$. Therefore, $t \leq ab$ so that $ab$ is the greatest lower bound of a and b. □

**Lemma 6.3.** *$L$ is a relatively complemented distributive lattice.*

*Proof.* From Lemmas (4) and (5) above follows that L is a lattice and $ab$ is the greatest lower bound of a and b; and also L is a distributive lattice. Now $a \leq b, 0 \leq x \leq a$. Put $y = a - x$. Clearly, $0 \leq y \leq a$. Further $y + x = (a-x) + x = a + x = a = y - [(y-x) + (x-y)] = [y - (x-y)] - (y-x) = y - (y-x) = y - [(a-x) - x] = y - (a-x) = 0$. Thus $[0, a]$ is complemented for any a in L so that L is relatively complemented. □

*Proof.* Proof pf Theorem(6.14) We have already observed in the above Lemma that L is relatively complemented distributive lattice with least element zero and hence L has the structure of a Boolean ring. □

Now the following Theorem shows that if $\star$ is a group operation on L, then L has the structure of a Boolean ring.

**Theorem 6.16.** *In L, the following statements are equivalent.*

*i) $\star$ is a group operation.*

*ii)* $\star$ *is associative.*

*iii)* $\star$ *is cancellative.*

*iv)* $a-(b-a)=a$ *for all* $a,b$ *in* $L$.

*v)* $(L,\star,\cdot)$ *is Boolean ring.*

*vi)* $(L,\star)$ *is a loop.*

*vii)* $ab$ *is the greatest lower bound of* $a$ *and* $b$ *in* $L$.

*viii)* $a+(bc)=(a+b)(a+e)$ *for all* $a,b,c$ *in* $L$.

*ix)* $a-(bc)=(a-b)+(a-e)$ *for all* $a,b,c$ *in* $L$.

*x)* $x\leq y\Rightarrow ax\leq ay$ *for all* $a$ *in* $L$.

*Proof.* The order of demonstration is:

a) $(i)\Rightarrow(ii),(iii),(vi)$

b) $(ii)\Rightarrow(i),(iii)\Rightarrow(iv)\Rightarrow(v)\Rightarrow(i),(vi)$

c) $(vi)\Rightarrow(iii),(vii)\Rightarrow(iv),(viii),(ix),(x)$

d) $(viii)\Rightarrow(iv);(ix)\Rightarrow(vii);(x)\Rightarrow(vii)$

Now $(i)\Rightarrow(ii),(iii)$ and $(vi)\Rightarrow(iv)$, $(iv)\Rightarrow(v)\Rightarrow(i)$, $(vi)\Rightarrow(iii)$, $(vii)\Rightarrow(viii),(ix),(x)$ are all obvious.
Assume (ii). Since $a\star a=0$ and $a\star 0=0\star a=a$ follows, $(ii)\Rightarrow(i)$.
Assume (iii) Then $a\star(b-a)=[a-(b-a)]+[(b-a)-a]=[b-(b-a)]+(b-a)=a+(b-a)=a+b$ so that $[a-(b-a)]+(b-a)=(b-a)\star[a-(b-a)]=(b-a)+a=a+b$ so that $a-(b-a)=a$. Hence (iii)$\Rightarrow$ (iv).
Assume (v): Let $ab$ be the greatest lower bound of a and b in L. Then since (v)$\Rightarrow$(i), we have $ab=a-(a\star b)=[a\star(a\star b)]$ (since in a Brouwerian ring $(B,+,\cdot)$, $a-b=a+ab$). So, $ab=a[(a\times a)\times(a\times b)]=a\star(a\times b)=(a\times a)\times(a\times b)=a\times b$. Hence $(v)\Rightarrow(vii)$.
Assume (vii): Let $a,b$ in L. Then $a=(a+b)a=a-[a\times(a+b)]=a-[(a+b)-a]=a-(b-a)$ so that (vii) $\Rightarrow$ (iv).
Assume (viii): Now $a=a+(ab)=(a+a)(a+b)=a(a+b)=a-(b-a)$ so that $(viii)\Rightarrow(iv)$.
Assume (ix): Let $a,b\in L$. We already know that ab is a lower bound of a and b. Let $t$ be a lower bound of a and b. Then $0=(t-a)+(t-b)=t-(ab)$ so that $t\leq ab$. Hence $(ix)\Rightarrow(vi)$.
Assume (x): Let $a,b\in L$ and $t$ be a lower bound of a and b. Now $t\leq a$, $t\leq b$ implies that $t=t^2\leq tb\leq ab$ so that $(x)\Rightarrow(vii)$. Thus the proof of the theorem is complete. □

Let $(L,+,0,-)$ be a semi-Brouwerian algebra. L is autometrized by the symmetric difference. No we have

**Theorem 6.17.** *L is a Boolean ring if and only if it is free of isosceles triangles.*

*Proof.* Proof of necessary is obvious.
Conversely, suppose that $a\star b=a\star c, b\neq c$. i.e. $a,b,c$ are vertices of an isosceles triangle. If L is free of isosceles triangles, then $\star$ is cancellative and therefore L is a Boolean ring. □

**Theorem 6.18.** *A semi- Brouwerian algebra is a Brouwerian ring if and only if it admits a metric group operation. We need the following two Lemmas*

**Lemma 6.4.** *Let $\theta$ be a metric group operation on a semi Brouwerian algebra L. Then for all $\theta$ in L, $a\theta b \leq a+b$.*

*Proof.* Since $0\theta 0 = 0$ and $a\theta 0 = a, a\theta b \leq (a\theta 0) + (0\theta b) = a + b$. □

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**Lemma 6.5.** *For any a,b in L,*

$$(a+b) \times \{a - [a - (b - a))]\} = a + b.$$

*Proof.* First we observe that $(a + b) - a = (a + b) - \{a - (b - a)\}$ for all a,b in L. Since $a - (b - a) \leq a$ we have $(a + b) - a \leq (a + b) - [a - (b - a)]$. Also $(a + b) - [a - (b - a)] \leq b - a = (a + b) - a$ Since $(b-a)+[a-(b-a)] = a+(b-a0 = a+b$. Thus $(a+b)-a = (a+b)-[a-(b-a)]$. Now putting $s = a-(b-a)$ and $t = (a+b)-(a-s)$ we have $t+(a-b) = (a+b)+(a-s) = a+b$ so that $a+b = t+(a-b) \leq t+[(a+b)-s] \leq a+b$. Hence

$$\begin{aligned} a + b &= t + [(a + b) - s] = t + [(a + b) - a] \\ &= t + [\{t + (a - s)\} - a] = t + (t - a) = t \\ &= (a + b) * \{a - [a(b - a)]\} \end{aligned}$$

□

*Proof.* of Theorem 2.4.2(ii). Suppose $\theta$ is a metric group operation in a semi Brouwerian algebra L and $c, d \in L$. Then $c = c\theta 0 \leq (c\theta d) + (d\theta 0) \leq c\theta d) + d$ so that $c - d \leq a\theta d$ and similarly $d - c \leq c\theta d$ so that $c\theta d \geq c * d$. Now let $a, b \in L$ and put $a + b = x$. By lemma (4) above,

$$x = x * \{a - [a - (b - a)]\} \leq x\theta\{a - (a - (b - a))\}$$

and so by applying Lemma(3) above, we get $x\theta\{a - [a - (b - a)]\} = x = x\theta 0$ so that $a - [a - (b - a)] = 0$. Hence $a = a - (b - a)$ so that L is a Boolean Ring. The converse is clear. □

# Betweenness

**Definition 6.1.** *Let $(L, +, 0, -)$ be a semi Brouwerian algebra. For $a, b, c$ in L, we say b lies between a and c (semi lattice theoretically)iff $ab + bc = b = (a + b)(b + c)$ (in symbols $(a, b, c)S$ (Here $xy = (x + y)_(x * y)$).*

Now we have

2.4.3(ii) $(a, b, c)S \Rightarrow ac \leq b \leq a + c$.

*Proof.* If $ab + bc = b = (a + b)(b + c)$, then, $b = ab + bc \leq a + c$; also

$$\begin{aligned}
&a - \{[(a - c) - b] + [(c - a) - b]\} \\
&= a - \{[a - (b + c)][c - (a + b)]\} \\
&\leq [(a + b) - \{a - (b + c)\} + [c - (a + b)]] \\
&= (a + b) - \{[(a + b) - (b + c)]\} + \{(b + c) - (a + b)\} \\
&= (a + b) - \{[(a + b) * (b + c)]\} = (a + b) \cdot (b + c) = b
\end{aligned}$$

So that $a \leq b+[(a-c)-b]+[(c-a)-b] \leq b+(a-c)+(c-a) = b+(a*c)$ and hence $ac = a-(a*c) \leq b$. □

2.4.3(iii) Theorem L is a Boolean geometry iff $ac \leq b \leq a + c$ implies $(a, b, c)S$.

*Proof.* The necessity is obvious in a distributive lattice $ac \leq b \leq a + c$ iff $(a, b, c)S$.
Conversely suppose that $ac \leq b \leq a + c$ implies $(a, b, c)S$. Hence by hypothesis, we have $a, b, b)S$ so that $b = (a + b)(b + b) = (a + b)b = (b - [a + b) - b] \leq b - (a - b)$. Hence by theorem (7) in 2.4.1 follows that L is a Boolean geometry. □

2.4.3(iv) Definition Let $(L, +, -, 0)$ be a semi Browerian algebra. We say $(L, *)$ is symmetric iff $(a+b)*(ab) = a*b$.

2.4.3(v) Theorem if $(a - b) + ab = a$ for all a,b in a semi Browerian algebra L, then

(1) L is symmetric and

(2) Semi lattice betweenness implies metric betweenness.

*Proof.* Since $a = (a-b)+ab$, we have $a-(ab) = [(a-b)+ab]-(ab) \leq a-b \leq a-(ab)$ so that $a-b = a-(ab)$. Hence $(a + b) * (ab) = a + b - (ab) = [a - (ab)] + [b - (ab)] = (a - b) + (b - a) = a * b$ i.e, L is symmetric. Assuming, $(a, b, c)S$ we have $ae \leq b \leq a + e$ (by Th 2.4.3(ii)). Now $ac \leq b \Rightarrow a - b \leq a - (ac) = a - c$. Also $b \leq a+c \Rightarrow b-c \leq a$ and $b-a \leq c$ so that $b-c = (b-c)-c \leq a-c$ and $b-a = (b-a)-a \leq a-a$. Now we have $a - b \leq a - c$, $b - a \leq c - a$, $c - b \leq c - a$ and $b - c \leq a - c$ so that $a * b \leq a * c$ and $b * c \leq a * c$ $(a * b) + (b * c) \leq a * c$ so that $(a * b) + (b * c) \leq a * c$ we, $(a, b, c)M$ holds. □

2.4.3(vi) Theorem L is a Boolean geometry iff semi lattice beweenness coincide with metric betweenness.

*Proof.* Suppose that in L semi lattice a betweenness coincides with metric betweeness. It is enough to show that r is cancellative (in view of Th. 2.4.1(I)) Now let $a, b, c \in L$ with $a * b = a * c$. Thus it follows that (a,b,c)M and (a,c,b)M from which we have $(a, b, c)S$ and $(a, c, b)S$. Hence $ab+bc = b$ and $ac+cb = c$ so that $ab = a - (a * b) = a - (a * c)ac$. Hence $b = ab + bc = ac + bc = c$
The converse part is a result in Boolean algebra which is true.

□

2.4.3(vii) Theorem L is a Boolean geometry iff metric betweenness has translation with $t_1$

*Proof.* we need only to show the sufficiency. Suppose that metric betweenness in L has $t_1$. Let $a, b, c \in L$ with $a * b = a * c$. It follows that $(a, b, c)M$ and $(a, c, b)M$ so that by $t_1$ we have $(c, b, c)M$. Hence $c * c = (c * b) + (b * c)$ so that $b * c = 0$. Hence and so by Th 2.4.(7) follwes that L is a Boolean geometry. □

2.4.3(viii) A semi Brouwerian algebra is a Boolean algebra iff its group of isometry is simply translative.

*Proof.* Necessity is Known. Sufficiency follows from Theorem 1.5.8 and from Theorem 2.4.1(7). □

# Chapter 3

# Lattice Ordered Groups

We briefly mention definition of lattice ordered groups and some properties for the sake of convenience some of the properties we may miss to include here, may be found in part I of this BUTR or in Biskhoffi lattice theorty. We consider only commutative lattice ordered Groups.

**Definition 0.1.** *An abelian group $(G, +, -, 0)$ is called a commutative lattice ordered group if $(G, \cup, \cap)$ is a lattice which satisfies the condition $x \leq y$ implies $a + x \leq a + y$ for any $a, x, y$ in $G$. Hence, $\leq$ is the partial order induced by the lattice operations $\cup$, $\cap$ on $G$.*

We denote this by $(G, +, -, 0, \cup, \cap)$ is an $\ell$-group or some times as $(G, +, -, 0, \leq)$ also. Also $a \leq b$ if and only if $a \cup b = b$ or equivalently $a \cap b = a$.
Examples of $\ell$- groups are plenty. For instance, $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$,i.e, the system of real numbers, system of rational numbers and the system of integers under usual addition $+$ and usual ordering $\leq$ are classical examples of $\ell$-groups. Also $\mathbb{R}^*$ the set of all nonnegative real numbers under multiplication and usual ordering also a group but it disobeys the law $a \leq b \implies ax \leq bx$. But the set of positive numbers under multiplication and usual ordering is an $\ell$-group.
We state below some consequences of the definition

**Theorem 0.1.** *Let $(G, +, -, 0, \cup, \cap)$ be an $\ell$-group, and $a, b, c$ be elements of $G$.*

*(i)* $a + (b \cup c) = (a + b) \cup (a + c)$

*(ii)* $a + (b \cap c) = (a + b) \cap (a + c)$

*(iii)* $a - (b \cap c) = (a - b) \cup (a - c)$

*(iv)* $a - (b \cup c) = (a - b) \cap (a - c)$

*(v)* $(b \cup c) - a = (b - a) \cup (c - a)$

*(vi)* $(b \cap c) - a = (b - a) \cap (c - a)$

**Theorem 0.2.** *For any $a$ in an $\ell$-group $G$, let $|a|$ denote $a \cup (-a)$. This $|a|$ is called the absolute of $a$.*

*(i)* $|a| \geq 0$ *with equality if and only if* $a = 0$

*(ii)* $|a| = |-a|$

*(iii)* $|a-c| \leq |a-b| + |b-c|$

*(iv)* $|a| + |b| \leq |a+b| \cup |a-b|$

*(v)* $||a| - |b|| \leq |a-b| \cap |a+b|$

*(vi)* $|a-b| = (a \cup b) - (a \cap b)$

*(vii)* $a + b = (a \cup b) + (a \cap b)$

*(viii)* $|nx| = |n||x|$ *for any integer* $n$ *and* $x$

*(ix)* $a \cap b = 0$, $a \cap c = 0 \implies a \cap (b+c) = 0$.

**Theorem 0.3.** *For any* $x, y$ *in an* $\ell$*-group* $(G, \cup, \cap, +, -, 0)$ $d(x,y)$ *denotes* $|x-y|$*, then,* $(G, G, d)$ *a metric OMM-space.*

*Proof.* We need to observe $(m_1)$ through $(m_4)$ are satisfied.
$d(x,y) = |x-y| \geq 0$ for any $x, y$ in $G$.
$d(x,x) = |x-x| = |0| = 0$ for any $x$ in $G$.
$d(x,y) = |x-y| = |-(x-y)| = |y-x| = d(y,x)$ for any $x, y$ in $G$.
$d(x,y) + d(y,z) = |x-y| + |y-z| = |x-y+y-z| \cup |x-y-y+z| = |x-z| \cup |x+z-2y| \geq |x-z|$ for any $x, y, z$ in $G$.
Also $|x-y| = 0 \implies x-y = 0 \implies x = y$. Thus $(G, G, d)$ is a metric OMM-space. □

**Definition 0.2.** *If the ordering in an* $\ell$*-group is total, then we say* $G$ *is a totally ordered group or a simply ordered group or a chain.*

Recall that from OMM-spaces, a bijection $\sigma : G \to G$ where $G$ is an $\ell$-group is called an isometry if and only if $|x-y| = |\sigma(x) - \sigma(y)|$ (=distance preserving property). In view of $(m_4)$, $\sigma$ is an isometry if and only if $\sigma$ is a surjection and distance preserving.
Any mapping $\sigma : G \to G$ with $|x-y| = |\sigma(x) - \sigma(y)|$ is called a congruence from $G$ to $G$.
Also for any $\ell$-group $G$, $G^*$ denotes the group of all isometries of $G$.
We first determine all the isometries of a totally ordered group.

**Theorem 0.4.** *The isometries of a totally ordered group* $(G, +, 0, -, \leq)$ *are the translations.*

*Proof.* Let $\sigma : G \to G$ be an isometry of a totally ordered group $G$. Let $\sigma(0) = a$. For any $x$ in $G$, $|\sigma(x) - \sigma(0)| = |x-0| = |x| = \pm x$ (as $G$ is a chain).
$\implies \sigma(x) - a = x$ or $\sigma(x) - a = -x$
$\implies \sigma(x) = x + a$ or $\sigma(x) = -x + a$
Suppose for all $x$, $\sigma(x) = x + a$, and for a $y$, $\sigma(y) = -y + a$. We may assume that $x$ and $y$ are different nonzero elements of $G$.
As $G$ is a chain, we may assume that $x < y$. Now $y - x = |y-x| = |\sigma(y) - \sigma(x)| = |a - y - (a+x)| = |-y-x| = |x+y|$

$\implies y - x = y + x$ or $y - x = -y - x$

$\implies x = 0$ in the first case and $y = 0$ in the second case. But one of these can happen. So for all $x$ in $G$, $\sigma(x) = x + a$ or for all $x$ in $G$, $\sigma(x) = -x + a$. Thus the theorem is proved.

Also observe that for any $a$, $x \mapsto a + x$ is an isometry and for any $a$ the mapping $x \mapsto -x + a$ is also an isometry. Hence, we conclude that all the isometries are given by these two forms. □

**Theorem 0.5.** *The set of all translations of an $\ell$-group forms a subgroup of the group $G^*$ of all isometries of $G$.*

*Proof.* If $\sigma_a(x) = x + a$, $\sigma_b(x) = x + b$, then $(\sigma_a o \sigma_b)(x) = \sigma_{a+b}(x)$ so that $\sigma_a o \sigma_b$ is $\sigma_{a+b}$ which is a translation. $\sigma_a^{-1}(x) = \sigma_{-a}(x)$ as $(\sigma_a o \sigma_{-a})x = 1(x) \implies (\sigma_a)^{-1} = \sigma_{-a}$. □

Note: Theorem 0.4 can be ordered lie direct product of a family of totally ordered groups.
For this we need definition

**Definition 0.3.** *If $S$ and $S'$ are subsets of an $\ell$-group $G$, a distance preserving map $\sigma : S \to S'$ is called a congruence form $S$ to $S'$.*

Now we have:

**Theorem 0.6.** *Let $G = \prod_{i \in I} G_i$ be the direct product of a non empty family of totally ordered groups $\{G_i | i \in I\}$. Let $S$ and $S'$ be subsets of $G$ and let $f : S \to S'$ be a congruence. Then $f$ is given by*

$$f(x)(i) = -a_i + (f(x))_i + x_i \ \forall x in \ S$$

*or*

$$f(x)(i) = a_i + (f(x))_i - x_i \ \forall x in \ S$$

*(Here $x_i, a_i$ denote the $i^{th}$ component of $x$ and $a$ respectively)*

*Proof.* For any $x \in S$, $|x - a| = |f(x) - f(a)| \implies |x - a|_i = |f(x) - f(a)|_i \implies |x_i - a_i| = |(f(x))_i - (f(a))_i|$ for any $i \in I$, as $a_i$ is totally ordered.

Suppose for $x$ in $S$, with $x_i \neq a_i$ and $y \in S$ with $y_i \neq a_i$,

$$(f(x))_i = (f(a))_i + x_i - a_i \text{and}$$

$$(f(y))_i = (f(a))_i + x_i - a_i.$$

Then, we get, $(f(x))_i - (f(y))_i = x_i + y_i - 2a_i$

$\implies |x - y|_i = x_i + y_i - 2a_i$

$\implies x_i - y_i = x_i + y_i - 2a_i$ or $x_i - y_i = -x_i - y_i + 2a_i$

$\implies x_i = a_i$ or $y_i = a_i$ which contradicts our assumption. Hence the result. □

**Lemma 0.1.** *If $\theta$ is a monomorphism from an $\ell$-group $G$ onto an $\ell$-group $H$ and if $f$ is an isometry of $G$, then, the mapping $f'$ defined by $f'(\theta(x)) = \theta(f(x))$ is a congruence from $\theta(G)$ onto $\theta(G)$.*

*Proof.* For any $x, y$ in $G$, $|x-y| = |f(x)-f(y)| \implies \theta(|x-y|) = \theta(|f(x)-f(y)|) \implies |\theta(x)-\theta(y)| = |\theta(f(x))-\theta(f(y))| = |f'(\theta(x))-f'(\theta(y))|$ □

Now we have the following theorem

**Theorem 0.7.** *If $f$ is an isometry of an $\ell$-group, then, there exists a unique involution isometric group automorphism $T$ of $G$ such that $f(x) = T(x) + f(0)$ for any $x$ in $G$.*

*Proof.* Since $G$ is a commutative $\ell$-group, by dliffords Theorem, $G$ is a subdirect product of a family of totally ordered groups $\{G_i | i \in I\}$. So those emith a monomorphism $\theta$ from $G$ into $\prod_{i\in I} G_i$.
Define $f'$ as in Lemma 0.1. Define $T'$ from $\theta(G)$ by $T'(\theta(x)) = f'(\theta(x)) - f'(\theta(0))$ for any $x$ in $G$. Let $S = S' = \theta(G)$ (as in Lemma 0.1), $a = \theta(0)(= 0\text{in}\prod_{i\in I} G_i)$.
For any $i \in I$, we get $(T'(\theta(x)))_i = (\theta(x))_i$ for every $x$ in $G$ or $(T'(\theta(x)))_i = -(\theta(x))_i$ for every $x$ in $G$. Hence, $T'$ is an involution group automorphism of $\theta(G)$. Put $T = \theta^{-1}oT'o\theta$. Since $f$ is an isometry, follows that $T$ is an involution isometry. □

**Corollary 0.1.** *If $f$ is an isometry of $G$, then $f$ can be written as $f = \tau oT$ where $\tau$ is a translation of $G$ and $T$ is an involutory isometric group automorphism.*

*Proof.* Let $\tau : x \mapsto x + f(0)$ and $T$ as in the above theorem. Then $f = \tau oT$. □

**Corollary 0.2.** *If $f$ is an isometry of $G$, then, $f^2$ is the translation $x \mapsto x + f^2(0)$.*

*Proof.* $f(x) = T(x) + f(0) \implies$
$f^2(x) = f(T(x) + f(0)) = T(T(x) + f(0)) + f(0) = T^2(x) + T(f(0)) + f(0)$
But $f^2(0) = f(f(0)) = T(f(0)) + f(0) \implies T(f(0)) = f^2(0) - f(0) \implies f^2(x) = x + f^2(0)$. □

**Corollary 0.3.** *The set of all translations of an $\ell$-group $G$ is a normal subgroup of the group of all isometries of $G$.*

*Proof.* Let $t$ be any translation of $G$ and $\sigma$ any isometry of $G$. $\sigma$ can be written as $\sigma = \tau oT$ where $\tau$ is a translation and $T$ is an involutory isometry of $G$ (by Corollary 0.1). We have, $\sigma oto\sigma^{-1} = \tau oTotoT^{-1}o\tau^{-1} = \tau oTotoTo\tau^{-1}$ which is a translation as $\tau$, $TotoT$, $\tau^{-1}$ are all translations. □

**Corollary 0.4.** *An isometry $f$ of $G$ is order preserving if and only if $f$ is a translation.*

*Proof.* If $f$ is a translation $x \mapsto x + a$, then by definition $\ell$-group, we have $x \le y \implies x + a \le y + a$ so that $f$ is order preserving.
Conversely assume $f$ is a order preserving isometry. We have $f(x) \le f(y) \implies f^2(x) \le f^2(y) \implies x \le y$ (by Corollary ??????). Thus $f$ is an order isomorphism and so we have $f(a \cup b) = f(a) \cup f(b)$, $f(a \cap b) = f(a) \cap f(b)$ for all $a, b$ in $G$. Now for any $x$ in $G$,
$x \cup 0 = |x \cup 0 - 0| = |f(x \cup 0) - f(0)| = f(x \cup 0) - f(0) \implies f(x \cup 0) = (x \cup 0) + f(0)$ and $-(x \cup 0) = 0 - (x \cap 0) = |0 - (x \cap 0)| = |f(0) - f(x \cap 0)| = f(0) - f(x \cap 0) \implies f(x \cap 0) = (x \cap 0) + f(0) \implies f(x) \cap f(0) = (x \cap 0) + f(0)$
so that $(x \cup 0) + f(0) + (x \cap 0) + f(0) = f(x) \cup f(0) + f(x) \cap f(0) = f(x) + f(0) \implies f(x) = x + f(0)$, i.e, $f$ is a translation. □

**Corollary 0.5.** *An isometry $f$ is order preserving if and only if $f$ is an involution of the form $x \mapsto a - x$.*

*Proof.* If $f$ is an order preserving isometry, then, $-f$ (defined by $(-f)(x) = -f(x)$ ) is an order preserving isometry and hence $-f$ is a translation $x \mapsto x + a$ so that $-f$ is the involution $x \mapsto -a - x$. Hence the theorem. □

**Theorem 0.8.** *If $G = G_1 \times G_2$ where $G, G_1, G_2$ are $\ell$-groups, then, the mapping $x \mapsto (x_1, -x_2)$ for any $x = (x_1, x_2)$ from $G$ into $G_1 \times G_2$ is an involution isometry fixing zero and also a group automorphism.*

*Proof.* Routine as all operations including the distance function are defined componentwise. □

Note: Isometries fixing zero are called stable isometries and also as 0-isometries.

**Theorem 0.9.** *Let $G$ be any $\ell$-group. For any involutory isometric group automorphism $T < G$, let $A = \{(T(x))^+ | x \in a^+\}$ and $B = \{(T(x))^- | x \in a^+\}$, $\bar{A} = A - A, \bar{B} = B - B$. Then, $G = \bar{A} \oplus \bar{B}$.*

*Proof.* Recall that for any $a$ in $G$, $a^+ = a \cup 0$, and $a^- = a \cap 0$.
Let $G$ be a subdirect product of a family of totally ordered groups $\{G_i | i \in I\}$. We have $(T(x))_0 = x_i$ for all $i \in I$ or $(T(x))_i = -x_i$ for all $i \in I$ as $T$ is involutory and fixes zero. We also have,

$$\begin{aligned}(T(x+y))_i &= x_i + y_i \ \forall i \in I \text{or} \\ &= -(x_i + y_i) \ \forall i \in I \text{ and}\end{aligned}$$

for any $x, y$ in $a^+$
$\implies (T(x+y))^+ = (T(x))^+ + (T(y))^+$ for all $x, y$ in $G$ and thus we have $A + A \subseteq A$. Also if $(T(x))^+ \geq y \geq 0$ in $G$, then $y_i \geq 0$ which implies $T(y)_i \neq -y_i$. Hence, $(t(y))_i = y_i$ so that $A$ is convex. A subset $S$ of a poset $P$ is said to be convex if $[a, b] \leq \infty$ whenever $a, b$ are in $S$. $[a, b]$ means the set of all $x \in P$ such that $a \leq x \leq b$.
Now $\bar{A}$ becomes a convex $\ell$- subgroup of $G$. Let $B = \{(T(x))^- | x \in G^+\}$ and $\bar{B} = B - B$. As in the case of $\bar{A}$, we get $\bar{B}$ is also a convex $\ell$-subgroup of $G$. Further $\bar{A} \cap \bar{B} = \{0\}$. If $y \in G^+$, then, $y_i = (\prod y)_i = (T(x))_i^+$, or $y_i = -(\prod y)_i = (T(x))_i^- \implies y = (T(y))^+ + (T(y))^- \subset \bar{A} + \bar{B}$. Thus $G = \bar{A} \oplus \bar{B}$. □

We have the converse of the above in the following

**Theorem 0.10.** *Let $G$ be an $\ell$-group and let $\bar{A}$, $\bar{B}$ be direct summands of $G$ with $G = \bar{A} \oplus \bar{B}$. Then there exists an involutory isometric group automorphism $T$ of $G$ such that $\bar{A}$ is the subgroup of $G$ generated by $\{T(x)^- | x \in G^+\}$*

*Proof.* Let $G = \bar{A} \oplus \bar{B}$. Then $x \in G$ can be uniquely written as $x = x_A + x_B$ where $x_A \in \bar{A}$, $x_B \in \bar{B}$. Define $T : G \to G$ by $T(x) = x_A - x_B$. Then $T$ is the required isometry. Now let $A = \{(T(x))^+ | x \in G^+\}$ and $B = \{(T(x))^- | x \in G^+\}$. Let $< A >= A - A$, $< B >= B - B$. If $x_A \in (\bar{A})^+, x_B \in (\bar{B})^T$, then $x_A = T(x_A) = (T(x_A))^+ \in A$ and $x_B = -T(x_B) = (T(x_B))^- \in B$. Thus $\bar{A} =< A >$ and $\bar{B} =< B >$. □

**Remark 0.1.** *All the above discussion shows that any isometry fixing zero is an involutory group automorphism and each isometries correspond in a one-one to the direct decompositions of $G$. This correspondence is established independently by Jakch, Holland and Pewell.*

Now we discuss to what extent that an $\ell$-group is determined by its group of isometries. For $\ell$-groups $G$ and $H$ below, let $G^*$ and $H^*$ denote the groups of isometries, respectively.

**Theorem 0.11.** *If $\theta$ is an isomorphism from $G$ into $H$, then, there is an order isomorphism from $G^*$ and $H^*$ carrying translations of $G$ onto translations of $H$.*

*Proof.* Note that $G^*$ is partially ordered if we define $f \leq g$ to mean $f(x) \leq g(x)$ for all $x$ in $G$. Similarly for $H$. For $f$ in $G^*$, define $f' : H \to H$ by $f'(\theta(x)) = \theta(f(x))$ for all $x$ in $G$. Then $f' \in H^*$ (can be verified in a routine means) as follows:

$$|f'(\theta(x)) - f'(\theta(y))| = |\theta(f(x)) - \theta(f(y))| = \theta(|f(x) - f(y)|) = \theta(|x - y|) = |Q(x) - Q(y)|$$

as $Q$ is an $\ell$-group isomorphism. By defining $Q^*$ by $Q^*(f) = f'$, we have verify $Q^* : G^* \to H^*$ is an order isomorphism. Further, if $f$ is a translation $x \mapsto x + a$, then, $Q^*(f)$ is the translation $Q(x) \mapsto Q(x) + Q(a)$. Since $Q^{-1}$ is also an Abelian $\ell$-group isomorphism $H$ to $G$, follows that any translation in $H$ is the image of some translation in $G$. □

Conversely, we have the following:

**Theorem 0.12.** *If $G$ and $H$ are $\ell$-groups such that there is an order isomorphism from $G^*$ and $H^*$, then, $G$ is isomorphic with $H$.*

*Proof.* Suppose $Q^* : G^* \to H^*$ is an order isomorphism. First we show that $Q^*$ carries translations of $G$ onto translations of $H$. For any $a$ in $G$, let $t_a$ denote the translation $x \mapsto x + a$. We have $Q^*(t_a) = \tau' o T'$ (by Corollary ??????) where $\tau'$ is a translation of $H$ and $T'$ an involutory group isomorphic automorphism of $H$.
Case (i): Assume $a \geq 0$. If $a = 0$, then, $t_a$ is the identity map of $Q$ and hence $Q^*(t_a)$ is the identity map of $H$.
Let $a > 0$(i.e, $a \leq 0$, $a \neq 0$) $\implies$ $t_a \geq i_a$ (where $i_G : G \to G$ is the identity map of $G$).
$\implies$ $Q^*(t_a) \leq i_H$
$\implies$ $T'(y') + a' \geq y'$ for all $y'$ in $H$. Assume $T' \neq i_H$. Let $H$ be embedded in a direct product (as a subdirect product) of a family $\{H_i | i \in I\}$ of totally ordered groups.
$T' \neq i_H$ $\implies$ $\exists$ at least one $i \in I$ such that $T'(y')_j = -y_j$ for all $y'$ in $H$. (as $T'$ is an involutory isometry $T'(y')_i = \pm y'_i$ ). Choose a $y'$ in $H$ so that $Q^*(t_a)$ is a translation of $H$.
Case (ii): Let $a < 0$ $\implies$ $t_a$ is a translation and inverse of $t_a$ $\implies$ $Q^*(t_{-a})$ is a translation in $H$ (by case (i)).
Case (iii): Let $a \in G$. We have $a = a^+ + a^-$
$\implies$ $Q^*(t_{a^+})$ and $Q^*(t_{a^-})$ are translations (by case (i) and (ii))
$\implies$ $Q^*(t_{a^+}) o Q^*(t_{a^-})$ is a translation in $H$. $\implies$ $Q^*(t_{a^+} o t_{a^-})$ is a translation in $H$
$\implies$ $Q^*(t_a)$ is a translation in $H$. □

For any $a$ in $G$, as $\theta^*(t_a) = t_{a'}$ for some $a'$, define $\theta : G \to H$ by $\theta(a) = a'$. For $a, b$ in $G$, we have

$$\theta^*(t_{a+b}) = \theta^*(t_a o t_b) = \theta^*(t_a) \leq \theta^*(t_b) = t_{a'} o t_{b'} = t_{a'+b'}$$

so that $\theta(a+b) = \theta(a) + \theta(b)$.
Also, $a \leq b \iff t_a = t_b \iff \theta^*(t_a) \leq \theta^*(t_b) \iff t_{a'} \leq t_{b'} \iff a' \leq b'$. Surjection of $\theta$ is immediate since for any $a'$ in $H$, $t_{a'} = \theta^*(t_a) \implies a' = \theta(a)$. Hence, $\theta$ is an automorphism from $G$ onto $H$.

**Remark 0.2.** *We remark here that the author ????? by assuming the hypothesis that an order isomorphism carries translations into translations that $G$ is isomorphic to $H$. Later Jauwtick improved that result and proved that any order isomorphism carries translations into translations. We have included here that improved version of Jauwtick.*

# 1 Lattice orderings and $f$-rings

**Definition 1.1.** *A system $(R, +, -, \cup, \cap, \cdot, 0)$ is called a lattice ordered ring (=$\ell$-ring) if*

*(1) $(A, +, -, \cup, \cap, 0)$ is an Abelian $\ell$-group.*

*(2) $(B, +, \cdot, 0, -)$ is a ring.*

*(3) $[(a \cup 0).(b \cup 0)] \cup 0 = 0$ for all $a, b$ in $R$.*

The condition (3) is equivalent to saying that the product of positive elements is positive. (Any $a \geq 0$ is called a positive element). If the multiplication $\cdot$ is commutative, we say the $\ell$-ring is commutative.
The following properties hold in any $\ell$-ring $R$

**Definition 1.2.**

$$a \geq 0 \implies a(b \cup c) \geq (ab) \cup (ac)$$
$$a(b \cap c) \leq (ab) \cap (ac)$$

$|ab| \leq |a||b|$

**Definition 1.3.** *A lattice ordered ring $R$ is called a $f$-ring (function ring ) if and only if $a \cap b = 0$, $c \leq 0 \implies a \cap (bc) = 0$. We consider only commutative $\ell$-rings and commutative $f$-rings.*

**Theorem 1.1.** *An $\ell$-ring $R$ is a $f$-ring if and only if :*

$$(a \cup 0) \cap [(-a \cup 0) \cdot (c \cup 0)] = 0 for\ any\ a, b, c in R$$

**Remark 1.1.** *(i) In any $f$-ring $a \cap b = 0$ implies $a \cdot b = 0$.*

*(ii) Any subdirectly irreducible $f$-ring is totally ordered.*

*(iii) Any $f$-ring is a subdirect product of totally ordered rings.*

*(iv) An $\ell$-ring $R$ is a $f$-ring if and only if $R$ is a subdirect product of totally ordered rings.*

*(v) In any $f$-ring the following hold:*
*If $a \geq 0$, then, $a \cdot (b \cup c) = (ab) \cup (ac)$, $a \cdot (b \cap c) = (ab) \cap (ac)$ and $|ab| \leq |a||b|$.*

*(vi) Any $\ell$-ring or a $f$-ring is an autometrized space (metric OMM-space) with distance function $d(a,b) = |a-b|$.*

*(vii) The following theorem determines all isometries of $f$-rings with unity.*

**Theorem 1.2.** *In a $f$-ring $R$ with unity, a mapping $f : R \to R$ is an isometry if and only if for $a, b$ in $R$ with $|a| = 1$, $f$ is given by $f(x) = ax + b$.*

*Proof.* Let $R$ be a $f$-ring with unity. Let $\theta$ be a monomorphism (of $f$-rings) from $R$ into a direct product $\prod_{i\in I} R_i$ where each $R_i$ is a totally ordered ring with unity (The embedding is as a subdirect product). Let $f : R \to R$ be an isometry of $R$. (This implies $f$ is an isometry of the additive group $(R, +, -, 0, \cup, \cap)$). Let $T, T', f'$ be as in ?????.

Now $T'$ is an isometry of $\theta(R)$ onto $\theta(R)$ given $i \in I$, we have either

$(T'(\theta(x)))_i = (\theta(x))_i$ for all $x$ or

$(T'(\theta(x)))_i = -(\theta(x))_i$ for all $x$ in $R$.

so that we can write $T'(\theta(x))_0 = a'_0(\theta(x))_i$ where $a'_0 = 1$ or $-1$; for all $x$ in $R$. Put $a' = T'(\theta(1))$, and observe that $a' \in \theta(R)$ so we can write $a' = \theta(a)$ for some $a$ in $R$.

Now $T'(\theta(x)) = \theta(a)\theta(x) = \theta(ax)$ for any $x$ so that $\theta(f(x)) = f'(\theta(x)) = T'(\theta(x)) + f'(\theta(0)) = \theta(ax + f(0))$ and hence $f(x) = ax + b$ for all $x$(Here $b = f(0)$). Also in any $f$-ring, 1 is positive and so we have $1 = |1| = |1 - 0| = |f(1) - f(0)| = |a|$.

For the other part of the theorem, let $a, b$ in $R$ with $a = 1$ and let $f$ be the mapping $f(x) = ax + b$.

For any $x, y$ in $R$, $|f(x) - f(y)| = |ax - ay| = |a(x - y)| = |a| \cdot |x - y| = |x - y|$ so that $f$ is distance preserving.

It remains to show that $f$ is a surjection.

$|a| = 1 \implies a^2 = 1$ by using the subdirect product representation of $R$ as a subdirect product totally ordered ring with unity (as in any totally ordered ring with unity $|a| = 1$ implies $a = 1$ or $a = -1$ so that $a^2 = 1$).

Now given $y$ in $R$, let $x = a(y - b)$. We have $f(x) = ax + b = a^2(y - b) + b = (y - b) = y$.

We are considering only abelian $\ell$-groups for metrization. The following theorem show that the need for choosing only abelian $\ell$-groups. □

**Theorem 1.3.** *Let $(G, +, \cup, \cap, -, 0)$ be any lattice ordered group (not necessarily abelian). If the function $d$ defined by $d(x, y) = |x - y|$ is a metric, then, $G$ is abelian.*

*Proof.* Assume that $d(x, y)$ is a metric. First we observe that $|x + y| \leq |x| + |y|$ for any $x$ and $y$. Since $|x + y| = |x - (-y)| = d(x, -y) \leq d(x, 0) + d(0, -y) = |x| + |y|$.

Case(i): Now assume $a \geq 0$, $b \geq 0$. Now

$$\begin{aligned} a + b &= |-(a + b)| (\text{as} |x| = |-x|) \\ &= |-b + (-a)| \leq |-b| + |-a| = |b| + |a| = b + a \end{aligned}$$

By symmetry $b + a \leq a + b$ so that $a + b = b + a$.

Case (ii): Let $a \geq 0$, $b \leq 0 \implies a \geq 0, -b \geq 0$

$\implies a + (-b) = (-b) + a$ by case (i)

$\implies b + a + (-b) = b + (-b) + a = 0 + a = a$

$\implies b + a + (-b) + b = a + b$

$\implies b + a = a + b$. Case (iii): By case (ii), we can say any negative element commutes with any positive element.

Now let $a \leq 0$, $b \leq 0$

$\implies -a \geq 0$, $-b \geq 0$

$\implies -a + (-b) = -b + (-a)$(by case (ii))

$\implies -(b + a) = -(a + b)$

$\implies b + a = a + b$

Case (iv): Let $a, b$ be any elements of $G$.

$$\begin{aligned} a + b &= a^+ + a^- + b^+ + b^- = a^- + a^+ + b^+ + b^- \\ &= a^+ + (b^+ + a^-) + b^- = b^+ + a^+ + a^- + b^- \\ &= b^+ + a^+ + b^- + a^- = b^+ + b^- + a^+ + a^- \\ &= b + a \end{aligned}$$

Thus $(G, +)$ is commutative. We have seen that $(m_2)$ the triangle inequality forces $+$ to be commutative. □

Note:There is another strengthening aspect for considering commutative $\ell$-groups. Holland explained that (by using a theorem of Ashwimirumar, the free lattice order) group on two or more generators have trivial centre and as $d(x, y)$ lies in the centre follows that $d(x, y) = 0$. Holland determining $d(x, y)$ as a word in the free group generated by two generalizations $x, y$ satisfying $d(a, b) = d(b, a)$, $d(a + c, b + c) = d(a, b)$ for $a, b, c$ (calling $d$ as a metric) shows that $d(x, y) = n|x - y|$ for some integer $n$. If $n = 1$, it is the usual metric. Also note that none of $m_3$ or $m_4$ are used in establishing this result.

The following shows that the metrization with $d(x, y) = |x - y|$ is almost unique.

**Theorem 1.4.** *Let $G$ be a commutative $\ell$-group and let $d : G \times G \to G$ be a mapping such that*

*(1)* $d(x, y) = d(y, x)$

*(2)* $d(x, z) \leq d(x, y) + d(y, z)$

*(3) Two mappings $x \mapsto a + x$, $x \mapsto a \cup x$ are contradictions*

*(4) $d$ is semiregular. Then $d(x, y) = |x - y|$.*

*Proof.* $d(a, b) = d(a + c - c, b + c - c) \leq d(a + c, b + c) \leq d(a, b)$ by (3)

$\implies d(a,b) = d(a+c, b+c)$. Now

$$\begin{aligned} d(a,b) &\leq d(a, a\cup b) + d(a\cup b, b) \\ &= d(a\cup b, a) + d(a\cup b, b) \\ &= d(a\cup b - a, a-a) + d(a\cup b - b, b-b) \\ &= d(a\cup b - a, 0) + d(a\cup b - b, 0) \\ &= (a\cup b - a) + (a\cup b - b) by (4) \\ &= (b-a)\cup 0 + (a-b)\cup 0 = |a-b| \end{aligned}$$

Thus $d(a,b) \leq |a-b|$.
Now $(a-b)\cup 0 = (a\cup b) - b = d((a\cup b)-b, 0) = d(a\cup b, b) = d(a\cup b, b\cup b) \leq d(a,b)$. Similarly, $(b-a)\cup 0 \leq d(b,a) = d(a,b)$.
$\implies (a-b)\cup(b-a)\cup 0 \leq d(a,b)$
$\implies (a-b)\cup(b-a) \leq d(a,b)$
Thus $d(a,b) = (a-b)\cup(b-a) = |a-b|$. □

**Remark 1.2.** *Note that Theorem ????, i.e., the theorem on uniqueness of metrization of representable autometrized algebras is also applicable to lattice ordered groups.*

# 2 Betweenness, segments and congruences

As any $\ell$-group is a distributive lattice and also a metric OMM-space, we can talk of lattice betweenness and metric betweenness. If happens that both the concepts coincide.

**Definition 2.1.** *We write $(a,b,c)L$ to mean $a\cap b \leq b \leq a\cup c$ and $(a,b,c)M$ to mean $(a*b)+(b*c) = a*c$.*

**Theorem 2.1.** *In autometrized $\ell$-group,lattice betweenness and metric betweenness are equivalent.*

*Proof.* Let $(a,x,b)L \implies a\cap b \leq x \leq a\cup b$
$\implies |a-b| - (a+b-2x) = (a\cup b) - (a\cap b) - (a\cup b) - (a\cap b) + 2x = 2(x - (a\cap b)) \geq 0$ so that $|a-b| \geq a+b-2x$.
Similarly, $|a-b| - (-a-b+2x) = 2((a\cup b) - x) \geq 0 \implies |a-b| \geq -(a+b-2x)$
$\implies |a-b| \geq (a+b-2x)\cup\{-(a+b-2x)\}$
$\implies |a-b| \geq |a+b-2x|$
$\implies |a-b| = |a-b| \cup |a+b-2x| = |a-x| + |x-b|$
$\implies (a,x,b)M$.
Now let us assume that $(a,x,b)M$,i.e., $|a-x| + |x-b| = |a-b| \implies |a-b| \geq a+b-2x$ and $-a-b+2x$ so that $2((a\cup b)-x) = |a-b| - (-a-b+2x) \geq 0$ and $2(x-(a\cap b)) = |a-b| - (a+b-2x) \geq 0$. Hence, $a\cap b \leq x \leq a\cup b$. □

**Definition 2.2.** *Given $a,b$ in a $\ell$-group $G$, the set of all $x$ between $a$ and $b$ is called the segment $[a,b]$.*

Now we observe that

**Theorem 2.2.** *Any two segments of equal length in any $\ell$-group $G$, are congruent.*

*Proof.* Let $[a, b]$ and $[c, d]$ be segments with $|a - b| = |c - d|$.
For any $x$ in $[a, b]$, define $\sigma(x) = x+(c\cap b)-(a\cap b)$ we have $|a-b| = |c-d| \implies (a\cup b)-(a\cap b) = (c\cup d)-(c\cap d) \implies (c\cap b)-(a\cap b) = (c\cup d)-(a\cup b)$. Since $a\cap b \le x \le a\cup b$ we have $\sigma(x)-(c\cap d) = x-(a\cap b) \ge 0 \implies c\cap d \le \sigma(x)$, Also $\sigma(x) = x + (c \cup d) - (a \cup b) \implies (c \cup d) - \sigma(x) = (a \cup b) - x \ge 0 \implies \sigma(x) \le c \cup d$. Thus $\sigma(x) \in [c, d]$. Also $|\sigma(x) - \sigma(y)| = |x - y|$ as $\sigma$ is a translation. This implies $\sigma$ is an injection. Also for $y \in [c, d]$, put $x = y - (c \cap d) + (a \cap b)$ implies $\sigma(x) = y$ and $x \in [a, b]$ so that $\sigma$ is a surjection. □

**Theorem 2.3.** *The congruent image of a segment is a segment of same length.*

For the proof, we need two lemmas.

**Lemma 2.1.** *If $x, y \in [0, a]$, then, $|x - y| = |x| + |y|$*

*Proof.* We have $a \cap 0 \le x, y \le a \cup 0$
$\implies x \cup (-y) \ge (a \cap 0) \cup \{-(a \cup 0)\} = (a \cap 0) - (a \cap -a) = 0 \cap (a \cup -a) = 0 \cap |a| = 0$, and similarly, $(-x) \cup y \ge 0$ so that $x \cap (-y) \le 0$. Thus $(x, 0, -y)L$ holds $\implies (x, 0, -y)M$ holds, i.e., $|x + y| = |x| + |y|$. □

**Corollary 2.1.** *If $x, y \in [0, a]$, then, $|x - y| = ||x| - |y||$*

*Proof.* We have

$$\begin{aligned} ||x| - |y|| &= |x - y| \cap |x + y| \text{(by?????)} \\ &= |x - y| \cap (|x| + |y|) \text{(by Lemma??)} \qquad\qquad = |x - y| \end{aligned}$$

□

**Lemma 2.2.** *Any translate of a segment is a segment of same length.*

*Proof.* Let $x \mapsto x + d$ be a translation from a segment $[a, b]$. The translate of $[a, b]$ under this mapping is the image of $[a, b]$ and so is $[a + d, b + d]$, as $a \cap b \le x \le a \cup b \iff (a + d) \cap (b + d) \le x + d \le (a + d) \cup (b + d)$. □

Proof of Theorem 2.2: Any segment $[a, b]$ can be treated as a segment $[c, d]$ where $c < d$ (???? $c = a \cap b$, $d = a \cup b$).
Also in view of the Lemma 2.2 above, it is enough to prove that any congruent image of $[0, a]$, where $a > 0$, is a segment of same length.
Let $\sigma : [0, a] \to S$ be a congruence. Let $y \in S$. Then there exists an $x$ in $[0, a]$ such that $y = \sigma(x)$.
$\implies |y - (0)| + |\sigma(0) - y| = |\sigma(a) - \sigma(0)|$ so that $y \in [\sigma(0), \sigma(a)] \implies S \subseteq [\sigma(0), \sigma(a)]$.
Now let $y \in [\sigma(0), \sigma(a)]$. Put $x = |y - \sigma(0)|$. Then $x \ge 0$ and since $a = |\sigma(a) - \sigma(0)| = |\sigma(a) - y| + |y - \sigma(0)|$
$\implies a - x = |\sigma(a) - y| \ge 0$ so that $x \in [0, a]$.
Now put $y_1 = \sigma(x)$ so that $y_1 \in S$. $\implies x = |x - 0| = |\sigma(x) - \sigma(0)| = |y_1 - \sigma(0)|$. Hence $0 = |y - \sigma(0)| - |y_1 - \sigma(0)| = |y - y_1|$ (by above corollary) as $y - \sigma(0)$ and $y_1 - \sigma(0)$ belong to $[0, \sigma(a) - \sigma(0)]$. Therefore $y = y_1 (= \sigma(x))$. Thus $[\sigma(0), \sigma(a)] \subseteq S$. Hence $S = [\sigma(0), \sigma(a)]$.

**Theorem 2.4.** *$\sigma(x) = |x-a|+(a\cap b)$ defines a congruence from $[a,b]$ to $[a,b]$ in which $\sigma(a) = a\cap b$, $\sigma(b) = a\cup b$.*

*Proof.* Clearly, $|\sigma(x)-\sigma(y)| = ||x-a|-|y-a|| = |x-y|$ (by Corollary of ????) so that $\sigma$ is distance preserving and hence one to one.
If $y \in [a,b]$, and if we put $x = |y-a| + (a\cap b)$, then, it is easily seen that $x \in [a,b]$ and since

$$|x-a| = ||y-a| + (a\cap b) - a| = ||y-a| - (a\cap b)| by Corollary?????? \qquad = y - (a\cap b)$$

follows that $\sigma(x) = y$. Thus $\sigma$ is a bijection and hence a congruence from $[a,b]$ to $[a,b]$. □

**Theorem 2.5.** *Any $\ell$-group $G$ does not contain an equilateral $\ell$-triangle.*

*Proof.* We have to prove that $G$ metrized by the distance function $d(x,y) = |x-y|$ does not contain equilateral $\ell$-triangles.
Let $\delta(a,b,c)$ be an equilateral triangle. Let $\alpha = |a-b| = |b-c| = |c-a|$.
$\implies 0 = ||a-b| - |b-c|| = |a-c| \cap |a+c-2b| = \alpha \cap |a+c-2b|$. Similarly $0 = \alpha \cap |b+a-2c|$
$\implies \alpha \cap \{|a+c-2b| + |b+c-2a|\} = 0$
$\implies \alpha \cap \{|a+c-2b+b+c-2a| \cup |a+c-2b-b-c+2a|\}$
$\implies \alpha \cap \{|2c-b-a| \cup |3(a-b)| = 0\}$
$\implies (\alpha \cap |2c-b-a|) \cup \alpha \cap (3d) = 0$
$\implies (\alpha \cap |2c-b-a|) \cup \alpha = 0$
$\implies \alpha = 0$ □

**Remark 2.1.** *Note that $G$ will however contain isosceles triangles where as Boolean geometry does not contain even isosceles traingles.*

**Definition 2.3.** *An $n$-tuple ($n \geq 3$) of distinct points is said to be $B$-linear if and only if there exists labelling $(p_i, p_j, p_k)M$ holds whenever $1 \leq i < j < k \leq n$; and is said to be $D$-linear if and only if there exists a labelling $(p_1, \cdots, p_n)$ of its elements such that $d(p_1, p_n) = \sum_{i=1}^{n-1} d(p_i, p_{i+1})$.*

Now we show that in $\ell$-groups, the notions of $B$-linearity and $D$-linearity are the same.

**Theorem 2.6.** *$B$-linearity and $D$-linearity are equivalent in $\ell$-groups.*

*Proof.* Suppose $P = \{p_1, p_2, \cdots, p_n\}$ is $B$-linear. Then there exists a labelling (say) $(p_1, p_3, \cdots, p_n)$ of its elements such that $(p_i, p_{i+1}, p_n)M$ for $i < n-1$ so that $|p_i - p_n| = |p_i - p_{i+1}| + |p_{i+1} - p_n|$. Hence $|p_i - p_n| = |p_1 - p_2| + |p_2 - p_n| = |p_1 - p_2| + |p_2 - p_3| + |p_3 - p_n| = \cdots = \sum_{i=1}^{n-1} |p_i - p_{i+1}|$ so that $p$ is $D$-linear.
Now suppose P is D-linear. Then, there exists a labeling $(p_1, p_2, \cdots, p_n)$ such that $|p_1 - p_n| = \sum_{i=1}^{n-1} |p_i - p_{i+1}|$ and $1 \leq i \leq s \leq n$. By hypothesis, it follows that

$$|p_s - p_i| = |\sum_{k=i}^{s-1} (p_k - p_{t+1})|$$

$$\leq \sum_{t=i}^{s-1} |p_t - p_{t+1}|$$

$$= |p_1 - p_n| - \sum_{i=1}^{s-1} |p_i - p_{t+1}| - \sum_{i=1}^{n-1} |p_i - p_{i+1}|$$

$$\leq |(p_1 - p_n) - \{\sum_{t=1}^{s-1}(p_t - p_{t+1}) - \sum_{t=s}^{n-1}(p_t - p_{t+1}\}| = |p_s - p_i|$$

and $p_i - p_s| = \sum_{t=0}^{s-1} |p_t - p_{t+1}|$.

Let $1 \leq i \leq j \leq s \leq n$; Then by the above, we have

$$|p_i - p_s| = \sum_{t=1}^{s-1} |p_t - p_{t+1}, |$$

$$|p_j - p_s| = \sum_{t=j}^{s-1} |p_t - p_{t+1}| \quad \textit{and}$$

$$|p_i - p_s| = \sum_{t=i}^{s-1} |p_t - p_{t+1}| \quad \textit{so that}$$

$$|p_i - p_j| + |p_j - p_s| = |p_i - p_s|.$$

Thus $(p_i, p_j, p_s)M$ holds and hence P is B-linear. □

# Chapter 4

# Drl-monoids

**4.1.0** We recall that a DRl-monoid is an algebra $(A, +, \vee, \wedge, -, 0)$ where

(i). $(A, +, \vee, \wedge, 0)$ is a commutative lattice ordered monoid (with $+$ as monoid operation) with 0 as an identity and will have $a + (b \wedge c = (a + b) \wedge (a + c), a + (b \vee c) = (a + b) \vee (a + c)$.

(ii). To each $a, b$ in $A$, there is a least $x$(denoted by $a - b$) lattice operators. i.e., $a \geq b \iff a = a \vee b \iff a \wedge b = b$).

(iii) .$(a - b) \vee 0 + b = a \vee b$.

**4.1.1** We now state below some consequences of the definition. Proofs may be found in part I.

(i). $a - a = 0$.

(ii). The condition (ii) in the definition can be replaced by the following identities:

(1). $(y - x) + x \geq y$.

(2).$x - y \leq (x \vee z) - y$.

(3).$(x + y) - y \leq y$.

(iii). $a - 0 = a$.

(iv). $a \leq b \implies a - c \leq b - c$.

(v). $a \leq b \iff a - b \leq 0$.

(vi). $a \leq b \implies c - b \leq c - a$.

(viii). $a - (b \wedge c) = (a - b) \vee (a - c)$

(ix). $(x - y) + (y - z) \geq x - y$.

(x). $a - (b + c) = (a - c) - b$.

(xi). If $a \geq b$, then, $(a - b) + b = a$

(xii). $(a - b) \vee 0 + (a \wedge b) = a$.

(xiii). $(a \vee b) - (a \wedge b) = (a - b) \vee (b - a)$.

(xiv). $a - (b - c) \leq a + (c - b)$.

(xv). $(a + b) - c \leq a + (b - c)$.

(xvi). $a \wedge b = 0 = a \wedge c \implies a \wedge (b + c) = 0$.

(xvii). $(a - b) \vee (b - a) + (a \wedge b) = a \vee b$.

(xviii) $.a \leq b \leq c \implies c - a = (c - b) + (b - a)$.

(xix). If $n$ is any positive intiger and $nx = 0$ then, $x = 0$ (where $nx$ defines as $x + x + x + ... + x$ with n-times).

(xx). The lattice $(A, \vee, \wedge)$ is a distributive lattice.

**4.1.1**$(i)$ **Theorem**: Let $(A, +, \vee, \wedge, -, 0)$ be a DRl-monoid. For any $x, y$ in $A$, let $d(x, y) = (x - y) \vee (y - x)$. Then, $(A, A, d)$ is a metric OMM-space.

*Proof.* $(a - b) \vee (b - a)$ is called the symmetric difference of $a$ and $b$. We also use $*$ in aplace of $d$ with in a set rotation. i.e., $a * b = (a - b) \vee (b - a)$. We show that $d$ satosfies $(m1)$ through $(m4)$. We have $a * b = (a - b) \vee (b - a) \geq 0$ (by 4.1(iii)). $a * a = (a - a) \vee (a - a) = 0.a * b = b * a$ is clear from the definition of $d$. $a * b = 0 \implies (a \vee b) - (a \wedge b) = 0 \implies a \vee a \wedge b \implies a = b$. Let $a, b, c \in A$. We have $(a - b) + (b - c) \geq a - c$ and $(c - b) + (b - c) \geq c - a$ (by 4.1.1 (ix)). Now, $(a * b) + (b * c) = (a - b) \vee (b - a) + (b - c) \vee (c - b) \geq (a - b) + (b - c) \geq (a - c)$, and $\geq (c - b) + (b - a) \geq c - a \implies (a * b) + (b * c) \geq (a - c) \vee (c - a) = a * c$. Thus, $(A, \wedge, *)$ is a metric OMM-space. □

**4.1.1(ii)**. Now we start with the following theorem:
**Theorem** Let $(A, +, \vee, \wedge, 0, -)$ be a totally ordered DRl-monoid. Then, either $A$ has a least element of $A$ is l-group.

*Proof.* Assume that $A$ has no least element. It has enough to show that every element in $A$ has a least element of $A$ is additively invertible. Let $a \in A$. If $a \leq 0$, then, $(0 - a) + a = 0$(by (xi) of 4.1.1) so that $a$ is invertible. Let $a > 0 \implies 0 - a \leq 0$. Since $A$ has no least, there ecits an element $x$ in $A$ such that $x < 0 - a \leq 0$. Since $A$ has least, there exits an element $x$ in $A$ such that $x < 0 - a \leq 0$. If $x + a > 0$, then, by definition weget, $0 - a \leq x$ which is not the case. So, $x + a \leq 0$. This implies that $x + a$ is invertible so that there is a $y$ in $A$ such that $x + a + y = 0$ which implies $a$ is invertible(with inverse $x + y$). Thus, $(A, +, \vee, \wedge, -, 0)$ is a l-group. □

**4.1.1(iii)**. $a * b = (a - b) \vee 0 + (b - a) \vee 0$

*Proof.* $a * b = (a - b) \vee (b - a) = (a \vee b) - (a \wedge b) = \{(a \vee b) - a\} + \{a - (a \wedge b)\}$ by $(xviii)$ of '4.1.1= $((b - a) \vee 0) + ((a - b) \vee 0)$. □

**4.1.1(iv)**. Any DRl-monoid is semiregular.

*Proof.* $a \geq 0 \implies a * 0 = (a \vee 0) - (a \wedge 0) = a - 0 = a$. □

**Remark:** We remark that every DRl-monoid need not be regular. But, every DRl-monoid with least element is regular.

**4.1.1(v):** A DRl-monoid is Boolean ring iff the symmetric difference is associative.

*Proof.* If $*$ denotes the symmetric difference, the meaning of the statement is a DRl-monoid $(A, +, \vee, \wedge, -, 0)$ is a Boolean Ring $(A, *, \wedge, 0)$ iff $*$ is associative.

We need the following lemma.

**Lemma:** In a DRl-monoid $(A, +, \vee, \wedge, -, 0)$ the following are equivalent.

1. $(a-b) \wedge (b-a) \leq 0$ for all $a, b$ in $A$.
2. $a - (b \vee c) = (a-b) \wedge (a-c)$ for all $a, b, c$ in $A$.
3. $(b \wedge c) - a = (b-a) \wedge (c-a)$ for all $a, b, c$ in $A$.

*Proof of Lemma.* Assume (1). Now, $b \wedge c \leq b, c \implies (b \wedge c) - a \leq (b-a) \wedge (c-a)$. Now, $\{(b-a) \wedge (c-a) -\}(b \wedge c) - a\} \leq [(b-a) - \{(b \wedge c) - a\}] \wedge [(c-a) - \{(b \wedge c) - a\}] = [(b - \{(b \wedge c)) - a\}] \wedge [c - \{(b \wedge c) - a\} - a] = [b - (\{(b \wedge c) - a\} + a)] \wedge [c - (\}(b \wedge c) - a) + a] \leq (b - (b \wedge a)) \wedge (c - (b \wedge a)) = (0 \vee (b-c)) \wedge (0 - (c-b))$ by $(viii)$ of 4.1.1= $0 \vee \{(b-a) \wedge (c-b)\} = 0$ by assumption (1). Thus, $(b-a) \wedge (c-a) \leq (b \wedge c) - a$. Hence, $b \wedge c) - a = (b-a) \wedge (c-a).i.e.(1) \implies (3)$. □

□

Now Assume (3). $((a-b) \wedge (b-a)) \vee 0 = (0 - (a \wedge b)) \wedge (0 - (b \wedge a)) = (a - (a \wedge b)) \wedge (b - (a \wedge b)) = (a-b) - (a-b) = 0$ by (3) $\implies (a \wedge b) \wedge (b-a) \leq 0$. Thus, (3) $\implies$ (4).

Now, Assume (1)

$\{(a-b) \wedge (a-c)\} - \{a - (b \vee c)\} \leq ((a-b)) - (a - (a - (b \vee c)) - b) \wedge (\{a - (a - (b \vee c)) - a) \leq ((b \vee c) - b) \wedge ((b \vee c) - c)$ by definition of $\leq (0 \vee (c-b)) \wedge (0 \vee (b-c)) = 0 \vee ((c-b) \wedge (b-c)) = 0$ by (1). Thus, (1) $\implies$ (2). Now assume (2)

$$\begin{aligned} \{(a-b) \wedge (b-a)\} \vee 0 &= \{(a-b) \vee 0\} \wedge \{(b-a) \vee 0\} \\ &= \{(a \vee b) - b\} \wedge \{(a \vee b) - a\} \\ &= (a \vee b) - (a \wedge b) \text{by assumption (2)} \\ &= 0 \implies (a-b) \wedge (b-a) \leq 0 \end{aligned}$$

. Thus, (2) $\implies$ (1).

Proof of the Theorem: By the associativityof $*$, we get for any $a$,

$$a * (a * 0) = (a * a) * 0 = 0 * 0 = 0 \implies a = a * 0$$

. i.e., $*$ is semiregular. Also, the regularity and commutativity of $*$ and the condition $a * a = 0$ give that $(A, *, 0)$ is an abelian group.

It remains only to show that $a \wedge (b * c) = (a \wedge b) * (a \wedge c)$ for all $a, b, c \in A$. We first observe that $a + a = a$.

$$\begin{aligned} a + a &= (a + a) * 0 \\ &= (a + a) * (a * a) \\ &= ((a + a) * a) * a \\ &= ((a + a) - a) * a \\ &= a - \{(a + a) - a\} \leq a \implies a + a \leq a \end{aligned}$$

so that $a + a = a$ (at every $a$ in $A$ is greater than zero). Now,

$$\begin{aligned} (a + b) - (a \vee b) &= (a + b) - \{(a \vee b) + (a \vee b)\} \\ &\leq \{a - (a \vee b)\} + \{b - (a \vee b)\} \\ &= 0 + 0 = 0 \implies a + b \leq a \vee b. \end{aligned}$$

But, $a \vee b \leq a + b$ as $a \geq 0, b \geq 0 \implies a + b = a \vee b$. Now,

$$\begin{aligned} (a - b) \wedge (b - a) &= (a - b) \wedge (b - a) * 0 \\ &= ((a - b) \wedge (b - a)) * ((a - b) \wedge (b - a)) \\ &= [(a - b) - \{(a - b) \wedge (b - a)\}] * (a - b) \\ &= \{(a - b) - (b - a)\} * (a - b) \\ &= \{a - (b + (b - a))\} * (a - b) \\ &= \{a - (b \vee (b - a)) * (a - b) \\ &= (a - b) * (a - b) \\ &= 0 \implies (a - b) \wedge (b - a) = 0. \end{aligned}$$

hence by the lemma, we have $a - (b \vee c) = (a - b) \wedge (b - a)$ and $(a \wedge b) - c = (a - c) \wedge (b - c)$. As we have $(x \vee y) * (x \wedge y) = x * y$, by letting $x = b - a, y = a$ we get $(a \vee (b - a)) * (a \wedge (b - a)) = a * (b - a) = (a \vee b) - (a \wedge (b - a)) = (b - a) \vee ((a \vee b) - (b - a)) = a \vee b \vee (b - a) = a \vee b$. Hence, $a \wedge (b - a) = 0$ for all $a, b$ in $A$. Now,

$$\begin{aligned} a \wedge (b - a) &= 0 \implies \{a \wedge (b - c)\} - \{(a \wedge b) - c\} \\ &= \{a - ((a \wedge b) - c)\} \wedge \{(b - a) - ((a \wedge b) - a)\} \\ &= (a - (a - c) \wedge (b - c)) \wedge ((b - c) - ((a - c) \wedge (b - c)))\} \\ &= (a - (a - c)) \wedge ((b - c) - (a - c)) \\ &= \{a - (a - c)\} \wedge \{(b - a) \wedge (b - (a \vee c))\} \leq a \wedge (b - a) \\ &= 0 \implies a \wedge (b - c) \leq (a \wedge b) - c. \end{aligned}$$

We always have, $(a \wedge b) - c \leq a \wedge (b - c)$. Hence, $(a \wedge b) - c = a \wedge (b - c)$. Now,

$$\begin{aligned} a \wedge (b * c) &= a \wedge ((b - c) \wedge (c - b)) \\ &= (a \wedge (b - c) \vee (a \wedge (c - b))) \\ &= ((a \wedge b) - c) \vee ((a \wedge c) - b) \\ &= (a \wedge (b - a)) \vee ((a \wedge b) - c) \vee ((a \wedge c) - c) \vee ((a \wedge b) - b) \\ &= (a \wedge b) * (a \wedge c). \end{aligned}$$

Hence follows that $(A, *, \wedge, 0)$ is a Boolean Ring.

**4.1.2** In a DRl-monoid with least element zero, we know by regularity and Th.1.5.4 that every isometry is of the form $x \to a * x$ for some $a$. But, for every $a$, the mapping $x \to x * a$ may not be isometry. The following is in the direction of determining those elements $a$ for which the mapping $a \to x * a$ is an isometry. First we considered totally ordered DRl-monoids.

**4.1.2(i)** Let $A = (A, +, \vee, \wedge, -, 0)$ be a totally ordered DRl-monoid with least element 0. Let $B = \{a \in A |$ the mapping $x \to a * x$ is an isometry$\}$. Then $(B, *, \wedge, 0)$ is a Boolean Ring.
For any $a \in A$,let $\sigma_a$ denote the mapping $x \rightarrowtail a * x$ as $A$ is totally ordered follows that $B$ is closed under $\vee$ and $\wedge$. Now, we show that $a, b \in B \implies a * b \in B$. As $a, b$ are in $B, \sigma_a, \sigma_b$ are isometries of $A$ and $\sigma_a \circ \sigma_b$ is an isometry of $A$. But, $\sigma_a \circ \sigma_b = \sigma_{a*b}$ so that $a * b \in B$.
Now we show that $a - b \in B$ whenever $a \in B, b \in B$. $a - b = (a - b) \vee 0 = a - (a \wedge b) = a * (a \wedge b)$ and so $a - b \in B$.
Now we show that $a + b \in B$ whenever $a, b$ are in $B$. such that $a \geq b$. Since $\sigma_a$ is surjection, then there exists an $x$ in $A$ such that $\sigma_a(x) = a + b$. i.e.,$a * x = a + b$. Now, $x = x * 0 = (a * x) * (a * 0) = (a * x) * a = (a + b) * a = (a + b) - a \leq b \leq a \implies a * x = a - x$. So, $a + b = a * x = a - x$. So that $a + b = a \in B$. So, $a + b \in B$ if $a \geq a$.. Hence, $(B, *, \wedge, 0)$nis a Boolean Ring as $B$ is also a sub-DRl-monoid in which $*$ is associative(as the set of isometries form a group under composition of mappings). The proof is complete.
The following lemma characterises the set $B$.
**Lemma:** Let $A = (A, +, \vee, \wedge, -, 0)$ be a totally ordered DRl-monoid with least element zero. Then, $B = \{a \in A | a - (a - y) = y$for all $y \leq a, a * y = y$ for all $y > a.\}$

*Proof of the lemma.* Write $d$ for the set on the right hand side. Now, let $a \in B$ so that $\sigma_a : x \rightarrowtail x * a$ is an isometry f $A$. Let $y \in A$. Since $\sigma_A$ is a surjection, then there exits an $x \in A$ such that $y \leq a$. Then, $x = x * 0 = (a * x) * (a * 0) = y * a = y - a \leq y$. If $x \leq a$, then, $y = a * x = a - y \leq a$, a contradiction to $x > a$. So, it follows that $y = a * x = x - a \leq x$ and $a * x = y$. This shows that $a \in d$. For the other part, let $a \in d$. We claim that $\sigma_a$ is an isometry. First we show that $\sigma_a$ is a surjection.

If $z \in A$, and if $z > a$, then, $\sigma_a(x) = a * z = z$ (since $a \in d$) and if $z \leq a$, then, $\sigma_{a-z} = a * (a - z) = a - (a - z) = y$ (as $a \in d$). Thus, $\sigma_a$ is a surjection.

□

Now, we show that $\sigma_a$ is distance preserving. For $x, y$ in $A$, since $A$ is totally ordered ,we have either of the following cases

(i). $0 \leq x \leq y \leq a$

(ii). $0 \leq a \leq x$ and $0 \leq a < y$.

(iii). $0 \leq x \leq a < y$.

*Proof.* Assume (i). Then,$a * x) * (a * y) = (a - x) * (a - y) = \{(a - x) - (a - y)\}$ and $x * y = y - x$. But, $[(a-x)-(a-y)]+x \geq a-(a-y) = y$ so that $y-x \leq (a-x)+(a-y)$ and $(a-x)+(a-y) = (a-y)+(y-x) \geq a-x$ so that $(a - x) - (a - y) = y - x$. i.e., $a * y) * (a * x) = y - x = x * y$.
Assume (ii) It is clear that $a * x) * (a * y) = x * y$.
Assume (iii). $a * x) * (a * y) = (a - x) * y = y - (a - x)$ (since $a - x \leq 0 \leq y$) and $x * y = y - x$.
But, $\{(a - x) - (a - y)\} + x = [a - (a - y) - x] + x \geq a - (a - y) = y$ so that $y - x \leq (a - y) + (y - x) \geq a - x$.
So that $(a - x) - (a - y) \leq y - x$.
Hence $(a-x)-(a-y) = y-x; (a*x)*(a*y) = x*y$ so that $\sigma_a$ is distance preserving. So, $d \leq B$. Hence, $B = d$.

□

**4.1.2(iii) Theorem:** Let $A = (A, +, \vee, \wedge, -, 0)$ be a totally ordered monoid with least element zero. Then, there exists at the most two isometries of $A$.

*Proof.* It is enough to show that the set $B$ has at the most two elements. Clearly $0 \in B$. Let $a > 0, b > 0, a \in B, b \in B$. Suppose $a > b$. then, $a = b * a = a - b$ so that

$$\begin{aligned} b = a - (a - b) &= a - (a - b) \text{ (as } a \in B) \\ &= a - a \\ &= 0. \end{aligned}$$

Which contradicts $a \neq 0$. So, $a \not\geq b$ similarly $b \not\geq a$ so $a = b$. □

**4.1.2(iv)Theorem:** Let $A, +, \vee, \wedge, -, 0)$ be a representable DRl-monoid with zer0 as least element. Let $B = \{a \in A | x \rightarrowtail a * x$is an isometry of $A\}$. Then, $(B, *, \wedge, 0)$ is a Boolean Ring.

*Proof.* As $A$ is representable DRl-monoid with least element, there exists a family $\{A_i | i \in I\}$ commutative totally ordered DRl-monoids each having a least element and that ...monomorphism $\theta : A \to \prod_{i \in I} A_i$ as a subdirect product. For the sake of simplicity in the presentation of proof and as there is no loss of generality, we may assume $\theta$ is the inclusion map from $A$ in to $\prod A_i$.
Suppose $\sigma : A \to A$ is an isometry. For any $i \in I$, define $\sigma_i : A_i \to A_i$ by $(\sigma(x))_i$ for any $x_i \in A_i$ and $x$ is an element of $A$write $x_i$ as the $i^th$ componet. We first show that $\sigma A_i$ is well defined.
Let $x_i \in A_i$ and let there be $x$ and $y$ in $A$ such that $x_i = y_i$. Enough to show that $(\sigma(x))_i = (\sigma(y))_i$. Since

$\sigma$ is distance preserving, we have $\sigma(x) * \sigma(y) = x * y \implies (\sigma(x))_i * (sigma(y))_i = x_i * y_i = 0 \implies (\sigma(x))_i = (\sigma(y))_i \implies \sigma_i$ is well defined.

Also, for any arbitrary $x_i$ and $y_i$ as $i^th$ components respectively, we have $\sigma_i(x_i) * \sigma_i(y_i = (\sigma(x))_i * (\sigma(y))_i = (\sigma(x) * \sigma(y))_i = (x * y)_i = x_i * y_i \implies \sigma(i)$ is distance preserving. $\sigma_i : A_i \to A_i$ is isometry and so $\sigma_i$ is the identity map of $A_i$ or $\sigma_i(x_i = a_i * x_i)$ for some $a_i > 0$(by 4.1.2(iii)).

In fact if $x \to a * x$ is an isometry of $A$ denoted by $\sigma_a$, then, for each $i \in I$, the mapping $\sigma_{a_i}(x_i = a_i * x_i)$ is an isometry of $A_i$. Now, it is only to observe that $B$ is closed under $*$. ie.,if $\sigma_a, \sigma_b$ are isometries then $\sigma_{a*b}$ is an isometry. Also, $*$ is associative. Now, we know that $B$ is closed under $\vee 4$ and $\wedge$.

Let $a, b \in B, i \in I$, then, $(((a \vee b) * x) * ((a \vee b) * y))_i = ((a \vee b) * x)_i * ((a \vee b) * y)_i = ((a_i \vee b_i) * x_i) * ((a_i \vee b_i) * y_i) = (a_i \vee b_i * x_i) * ((a_i \vee b_i) * y_i) = (a_i * x_i) * (a_i * y_i) = x_i * y_i$ or $(b_i * x_i) * (b_i * y_i) = x_i * y_i \implies \sigma_{a_i}, \sigma_{b_i}$ are isometries. Then, for each $i \in I, [(a \vee b) * x] * ((a \vee b) * y)_i = (x * y)_i \implies \sigma_{a \vee b}$ is distance preserving.

Now, let $y \in A$. We have $y = y * 0 = 0 * y = [(a \vee b) * 0] * [(a \vee b) * y] = (a \vee b) * [(a \vee b) * y]$ which shows that $\sigma_{a \vee b}((a \vee b) * y) = y$ so that $\sigma_{a \vee b}$ is surjection. Similarly, we can show that $\sigma_{a \wedge b}$ an isometry as $a * (a \wedge b) = a - b$ follows that $B$ is closed under $-$ also. □

It now follows that $(B, *, \wedge, 0)$ is Boolean Ring.

**4.1.3(i)Theorem.** Let $(A < +, \vee, \wedge, -, 0)$ be any DRl-monoid with least element zero. Then,

(1). Identity map is the only order preserving isometry.

(2). If $x \rightarrowtail a * x$ is order preserving isometr, then $a$ is greatest element of $A$.

*Proof(1).* Let $x \rightarrowtail a * x$ be order preserving isometry. $a = a * 0 = a * a = 0 \implies a = 0$ so that $x \rightarrowtail a * x$ is the identity mapping of $A$. □

*proof of (2).* Let $x \rightarrowtail a * x$ be an order reversing isometry of $A$, $0 \leq x \implies a = a * 0 \geq a * x$. Now, $x = x * 0 = (a * x) * (a * 0) = a - (a * x) \leq a - 0 = a$. As this is true for any $x$, follows that $a$ is the greatest element of $A$.

□

**4.1.3(ii).** Recall that any DRl-monoid $A$ is the direct product $A = D \times G$ where $D$ is aDRl-monoid with least element zero and $G$ is a l-group (Th,2,6,3(1) of part(I)). EVery isometry $\sigma$ of $A$ can be written as $\sigma = (\sigma_1, \sigma_2)$ when $\sigma_1$ is isometry of $D$ and $\sigma_2$ is isometry of $G$ (By Th 1.3.4) Now we have:

**4.1.3(iii).** Let $(A, +, \vee, \wedge, -, 0)$ be any DRl-monoid. Every isometry $\sigma$ of $A$ can be written as$\sigma = \tau \circ T$ where $T$ is an involuntary isometric monoid automorphism and $\tau$ is an isometry of $A$.

*Proof.* Let $A = +, \vee, \wedge, -, 0)$ be any DRl-monoid, $\sigma$isometry of $A$. We can write $\sigma = (\sigma_1, \sigma_2)$ where $\sigma_1$ is isometry of $D$ and $\sigma_2$ is isometry of $G$. By cor 3.1.5 (vii), $\sigma_2$ can be written as $\sigma_2 = \tau_2 \circ T_2$ where $T_2$ is an involuntary, isometric group automorphism of $G$, (and fixing zero ), and $\tau_2$ atranslation of $G$. As $D$ is regular, $\sigma_1$ is the form $x \rightarrowtail a * x$. Now, we write $\tau_1 = \sigma_1, T_1 = i_D$ (identity map of $D$. Then, $\sigma = (\sigma_1, \sigma_2 = (\tau_1 \circ T_1, \tau_2 \circ T_2)))$ Now we write $\tau = (\tau_1, \tau_2)$ and $T = (T_1, T_2)$. This gives $\sigma = \tau \circ T$.

$T$ is an involuntary, isometric, monoid automorphism because $T_1$ and $T_2$ are so of $D$ and $G$ respectively. □

**4.1.3(iv)**: Let $(A, +, \vee, \wedge, -, 0)$ be a DRl-monoid. Then, every isometry $\sigma$ of $A$ fiexed by zero is an involuntary monoid automorphism.

*Proof.* Let $A = D \times G, \sigma_1 = (\tau_1 \circ T_1), \sigma_2 = \tau_2 \circ T_2, \sigma = (\sigma_1, \sigma_2)$ as in Theorem 4.1.3(iii). Hence $T_1$ is the identity of $D$ and $\tau_1 = \sigma_1$.
$\tau_1(0) = 0 \implies \sigma 1(x) = \sigma_1(0) * x = \tau_1(0) * x = 0 * x = x \implies \sigma_1 = i_D = T_1 \implies \sigma_1$ is involuntary monoid automorphism.

□

**4.1.3(v)** Theorem: Let $A = (A, +, \vee, \wedge, -, 0)$ be DRl-monoid. Then, the isometries of $A$ fixing zero correspond one to oneto the direct summand of $A$.

*Proof.* Let $A = D \times G$ (as above), $\sigma : A \to A$ be an isometry fixing zero, $\sigma = (\sigma_1, \sigma_2)$ are isometries of $D$ and $G$ respectively, $\sigma(0) = 0 \implies \sigma_2(0) = 0 \implies \sigma_2$ is an isometric involunatry group automorphism of $G$.
By Th 3.1.6 (i) $G$ can be written as $G_1 \times G_2$ where $G_1$ and $G_2$ are l-groups of $G$. i.e., sub DRl-monoid of $A$. So $A = D \times G = (D \times G_1) \times G_2, D \times G_1$ is direct summand of $A$.
For the converse part, first observe that if $A = D \times G$ and if $\sigma_1, \sigma_2$ are isometries of $D$ and $G$ respectively, then, $x \rightarrowtail (\sigma_1(x), \sigma_2(x)), x \rightarrowtail (\sigma_1(x), -\sigma_2(x))$ are isometries of $A$. Now if $A_1$ is a direct summand of $A$, then, $A = A_1 \times A_2$ and $A_1 = D_1 \times G_1, A_2 = D_2 \times G_2$ gives the result as $A = (D_1 \times D_2 \times (G_1 \times G_2))$. □

**4.1.3(vi)**: The isometries fixing zero of a DRl-monoid depend only on the set of additively invertible elements.

*Proof.* Let $(A, +, \vee, \wedge, -, 0)$ be a DRl-monoid and let $\sigma : A \to A$ be an isometry, $\sigma(0) = 0 \implies \sigma * (i_D, \sigma_2)$ where $i_D$ is the identity map of $D$ and $\sigma_2$ is an isometry of $G$ which is involuntary, isometric group automorphism.
Conversely, any involuntary, isometric group automorphism $\sigma_2$ of $G$ determines an isometry $\sigma = (i_D, \sigma_2)$ of $A$ fixing zero, depend only on the isometries of $G$, the set of all invertible elements of $A$.. □

**4.1.3(vii) Theorem**: Let $f$ be an isometry of a DRl-monoid $A$ and let $a$ be an additively invertible invertibile element of $A$. Then, the mapping $g : A \to A$ defined by $g(x) = f(x) - a$ is an isometry of $A$.

*Proof.* We first observe that $g$ is surjection. Let $z \in A$. $\exists x \in A \implies f(x) = z + a$ as $f$ is an isometry $\implies f(x) - a = z.i.e., f(x) = z$. □

**4.1.3(viii) Theorem.** A DRl-monoid is l-group iff all the translations of $A$ (mapping of the form $x \to x + a$ ) are isometries.

*Proof.* If every mapping $x \to x + a$ is an isometry implies by surjectivity,$a$ is invertible and hence $A$ is l-group. The converse part is a result on l-groups, which is true. □

**4.1.3(vi). Theorem:** The isometries fixing zero of a DRl-monoid depend only on the set of additively invertible elements.

*Proof.* Let $(A, +, \vee, \wedge, -, 0)$ be a DRl-monoid and let $\sigma : A \to A$ be an isometry, $\sigma(0) = 0$ write $\sigma(0) = 0 \implies \sigma = (i_D, \sigma_2)$ where $i_D$ is the identity map of $D$ and $\sigma_2$ is an isometry of $G$ which is an involuntary group automorphism.
Conversely, any involuntary , isometric group automorphism $\sigma_2$ of $G$ determines an isometry $\sigma = (i_D, \sigma_2$ of $A$. Hence isometries of $A$ fixing zero, depened only on the isometries of $G$, the set of all invertible elements of $A$. □

**4.1.3(vii) Theorem.** A DRl-monoid is a l-group iff all the translations of $A$(mapping of the form $x \to x + a$) are isometries.

*Proof.* If every mapping $x \to x + a$ is an isometry implies by surjectivity, $a$ is invertble and hence $A$ is l-group. The converse part is a result on l-groups, which is true.. □

**4.1.4:** Uniqueness of Metric
**Theorem:** Let $A = (A, +, \vee, \wedge, -, 0$ be a DRl-monoid and $d : A \times A \to A$ be a mapping such that

(1 $d(a, b) = d(b, a)$.

(2) $d(a, b) + a \wedge b = a \vee b$.

(3) $d(a, c) \leq d(a, b) + d(b, c)$

(iv) $a \geq b \implies d(a, b) \leq a - b$.

Then, $d(a, b) = a * b$ (symmetric difference $(a - b) \vee (b - a)$.)

*Proof.* By (2) $d$ is semiregular for $a \geq 0 \implies d(a, 0) + a \wedge 0 = a \vee 0 \implies d(a, 0) = a$. By (2), $d(a, b) + (a \wedge b) = a \vee b \geq a \implies a - (a \wedge b) \leq d(a, b) \implies d(a, b) \geq 0$ as $(a - (a \wedge b) \geq 0$. $d(a, b) + b \geq d(a, b) + (a \wedge b) = a \vee b \geq a \implies a - b \leq d(a, b)$. Similarly, $b - a \leq d(a, b)$ so that $(a - b) \vee (b - a) \leq d(a, b)$. Now, $d(a, b) \leq d(a, a \vee b) + d(a \vee b, b) \leq ((a \vee b) - a) + ((a \vee b) - b) = (b - a) \vee 0 + (a - b) \vee 0 = a * b$. Hence, $d(a, b) = a * b$. □

Note: This result is applicable to l-groups, Boolean algebra and Brouwerian ALgebra. If we replace condition (2) above by the condition $(2)'$; $a \geq b \implies d(a, b) + b = a$ then, also the theorem is true and moreover it is applicable to Brouwerian algebras also.
Theorem 1.6.2 IS applicable to DRl-monoids.

**4.1.5. DRL-monoids with Unity**

An algebra $A = (A, +, \vee, \wedge, -, 0, 1)$ is called a DRl-monoid with unity if

(1). $(A, +, \vee, \wedge, 0, -)$ is a DRL-monoid.

(2). 1 is an element of $A$ such that $a + (1 - a) = 1 + 1$ for all $a$ in $A$.

Note that from the definition (1), it follows that $1 + 1 = 1$ (by holding $a = 0$. A l-group $(G, +, \vee, \wedge, 0)$ is also a DRl-monoid with unity if $1 = 0$. Boolean algebras and Brouwerian algebras are also DRl-monoids with unity $1 \neq 0$.

**4.1.5(i) Theorem** A DRl-monoid with unity is the direct product of a Boolean algebra and l-group iff $a - (b +$

$b) = (a - b) + (0 - b)$ and $1 - (1 - a) = a$ for all $a$ in $A$. (This is theorem 2.6.7(iii) of part I).

**4.1.5(ii)** Let $A$ be a DRl-monoid with unity of $a \vee (1 - a) \geq 1$, then, $A$ is Boolean algebra iff $A$ is regular and lattice betweeness is equivalent to metric betweeness.

*Proof.* Assume that $A$ is regular and lattice betweeness is equivalent to metric betweeness. Now, it is enough to show that $a$ in $A$ is complemented. Regularly implies $0 \leq a \leq 1$ for all $a$. Let $a \in A$, set $x = a * 1$ and $y = x * 1$. We have $x + y = x + (1 * x) = x + (1 - x) = 1$; and $a \vee x = a \vee (a * 1) = a \vee (1 - a) = 1 \implies (a, 1, x)L$ holds. Hence $(a, 1, x)M$ holds $\implies x + y = a * x$. Also, $a * x = (a \vee x) * (a \wedge x) = 1 * (a \wedge x) = 1 \implies (a \wedge x, 0, 1)M$ holds $\implies (a \wedge x, 0, 1)L$ holds $\implies a \wedge x \leq 0 \implies a \wedge x = 0$. Hence $(A, \vee, \wedge, ', 0, 1)$ is Boolean Algebra. The other part is a true result in Boolean algebra. □

**4.1.5(iii):**A DRl-monoid with unity if and only if the distance is invariant under all translations.

*Proof.* $((1 - 1) + 1) * (1 + 1) = 1 \implies 1 = 0$. The rest is obvious.

□

**4.1.5(iv) Theorem.** Any DRl-monoid $(A, +, \vee, \wedge, -, 0)$ is a Boolean Ring $(A, *, \wedge, 0$ iff

(i). $*$ is regular.

(ii). any first distance triangle has fixty.

*Proof.* Assume (i) and (ii). Enough to show that $*$ is associative(By Theorem 4.1.1(v)). Let $a.b, c \in A$. If any one of them is zero by regularity follows that $a * (b * c) = (a * b) * c$. For non zero, distinct $a, b, c$ the triangle $\triangle(0, a, b)$ and hence follows that $(a * b) * b = a$. So, the associative law is valid of any two of $a, b, c$ are equals. Now, the triangle $\triangle(a*b, a*(b*c), b*(b*c))$is the first distance triangle of $\triangle(a, b, b*c)$ and hence by hypothesis follows that $a * (b * c) = (a * b) * c$. The other part of $A$ is a result true in Boolean Algebras. □

**4.1.5(ii)** If all triangles are issosceles , then, $A$ is chain.

*Proof.* For any $a, b, \triangle(a \vee b, a \wedge b, a)$ is isosceles which implies $a$ and $b$ are comparable.

**4.1.5 (iii)** If $a \vee (1 - a) \geq 1$ in a DRl-monoid $A$ with unity, then, the following are equivalent.

(1) $A$ is free of isosceles trianles

(2) $A$ is regular and the group of isometries is simply transitive.

(3) $A$ is regular and the mapping $x \to 1 - x$ is an isometry.

(iv) $(A, \vee, \wedge, ', 0, 1)$ is Boolean Algebra.

We show that (1) $\implies$ (4) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (1). Assume (1) for any $a$, the triangle $\triangle(a, a*0, 0)$ is isosceles and hence $a*0 = a$ so that $a]geq0$. Since $a+(1-a) = a$ follows that $a \leq 1$. Hence, we have $a \vee (1-a)$. Now, for the isosceles triangle $\triangle(a \wedge (1 - a), 0, 1)$ we conclude that $a \wedge (1 - a) = 0$ so that $(A, \vee, \wedge, ', 0, 1)$ is a Booleanalgebra. Thus (1) $\implies$ (4).

(4) $\implies$ (2) is clear as is it is a result in Boolean algebras which is true.
Assume (2), since $*$ is regular, we get $a \vee (1-a) = 1$. Let $\sigma$ be the isometry which sends 1 to 0.
Let $\sigma(x) = y$. Then, $y = \sigma(x) * 0 = \sigma(x) * \sigma(1) = x * 1 = 1 - x$ so that $x \to 1 - x$ is an isometry. Thus (2) $\to$ (3).
Assume (3) since the mapping $x \to 1 - x$ is an injection we have $1 * 0 = 1 = 1 - (a \wedge (1-a)) = 1 * (a \wedge (1-a)) \implies 0 = a \wedge (1-a)$ so that a is complemented. Thus, (3) $\implies$ (4), (4) $\implies$ (4) is a result which is true in Boolean algebras. □

**4.1.5(iv):** Let $R$ be any betweeness relation , recall that we say for $a, b, c$ special inner point property holds if $(a, b, c)R$ and $(a, c, b)R \implies b = c$. We say $R$ satisfies transitivity $t_1 : (a, b, c)R$ and $(a, d, b)R \implies (d, b, c)R$. We say $R$ satisfies $t_2 : (a, b, c)R$ and $(a, d, b)R \implies (a, d, c)R$.
Note that metric betweeness relation in DRl-monoids fail to have $t_1$. For instance if $a, b, c$ are the vertices of an isosceles triiangle in a Brouwerian algebra $t_1$ fails.

**4.1.5(v):** In any DRl-monoid, $a, b, c$ fail to have special inner property if $a, b, c$ are the vertices of an isosceles traingle in which the sum of any two sides is their supremum.

*Proof.* See also 1.4.5(ii)of OMM spaces. Let $a, b, c$ be the vertices of an isosceles triangle and let $(a*b) \vee (a*c) = (a*b) = (a*c), (a*c) \vee (b*c) = (a*c) + (b*c)4$ and let $a * b = a * c$. Then, $a * c = (a * b) + (b * c) \implies (a, b, c)M$ and $b * c = (a * c) + (a * b) = a * c \implies a * b = (a * b) \vee (b * c) = (a * c) + (b * c) = (a * c) + (c*)$ so that $(a, c, b)M$ so special inner property fails. □

**4.1.5(vi):** If in a DRl-monoid, the metric betweeness has $t_1$, then, $A$ is free of isosceles triangles for which the sum of any two sides is equal to their union $\vee$.

*Proof.* If $\triangle(a, b, c)$ is such a triangle, where $a * b = a * c$ then, $(a, b, c)M$ and $(a, c, b)M$ hold. Hence by $t_1$ follows that $(c, b, c)M$ ie., $b = c$ which is not true. □

**4.1.5(vii):** In a DRl-monoid, metric betweeness has $t_2$.

*Proof.* Follows from Thm. 1.4.3 for OMM-spaces. □

**4.1.5(viii):** Theorem In any DRl-monoid, lattice betweeness and metric betweeness are equivalent iff metric betweeness has $t_1$

*Proof.* The proof depends on the following theorem of pitcher and Smiley □

**Theorem:** If $L$ is lattice and $R$ is a triadic relation on $L$, then, $R$ is a lattice betweeness relation provided the following conditions hold.

(1) $R$ satisfies the following conditions $\alpha$ and ($\beta$.

($\alpha$) : $(a, b, c)R \implies (c, b, a)R$

($\beta$ :) : $(a, b, c)R$ and $(a, c, b)R$ iff $b = c$.

(3) $R$ has $t_1$.

(4) $(a, a \vee c, c)R$ and $(a, a \wedge c, c)R$ hold for any $a$ and $c$.

(5) If $(a, b, c)R$, then, in the sublattice generated $a, b, c$, the transitivity $t_2$ holds for $R$.

*Proof of theorem 4.1.5(viii).* Assume the Hypothesis. It is enough to show that metric betweeness has the properties (1) through (5) of Pitcher and Smiley theorem. $M$ has $t_1, (a, b, c)M$ and $(a, c, b)M \implies b = c$. So, $(1), (2)$ and $(3)$ are satisfied.
$a * c = (a - c) \vee (c - a) = (a - c) \vee 0 + (c - a) \vee 0 = (a \vee c - a) + (a \vee c - c) = a * (a \vee c) + (a \vee c) * c$ also $a * c = (a * (a \wedge c)) + ((a \wedge c) * c)$ So, (4) is satisfied. Since metric betweeness has $t_2$, follows that all the five are satisfied. Hence metric betweeness is lattice betweeness.
Converse part is obvious because lattice betweeness has $t_1$. □

**4.1.6**: Let $\mathbb{D}$ be the variety of all DRl-monoids with unity. They form a variety because they are closed under the formation of subalgebras, homomorphic images and direct products. Let $\mathbb{B}$ be the variety of all Boolean algebras and $\mathbb{G}$ be the variety of all commutative l-groups. $\mathbb{B}$ and $\mathbb{G}$ can be treated as subvarieties of $\mathbb{D}$. Let $\mathbb{D}_{\nvdash}$ be the class of all algebras $\mathbb{D}$ in $\mathbb{D}$ which further satisfies the identities $(1-(1-a)) = a$ and $a-(b+b) = (a-b)+(0-b)$. $\mathbb{G}$ is subvariety of $\mathbb{D}$ and $\mathbb{B} \subset \mathbb{D}_{\nvdash}, \mathbb{G} \subset \mathbb{D}_{\nvdash}$. If $\mathbb{D}'$ is a subvariety of $D$ containing both $\mathbb{B}$ and $\mathbb{G}$ then $\mathbb{D}_{\nvdash} = \mathbb{D}'$. Thus, $\mathbb{D}_{\nvdash}$ is the least upper bound of the varieties $\mathbb{B}$ and $\mathbb{G}$.Hence, DRl-monoids with unity satisfying the additional identity is one of the best common abstraction of lattice ordered groups and Boolean algebras. We conclude this section by stating a Theorem of Prabhakara rao an DRl-monoids.
**Theorem**: Any DRl-monoid $A$ is isomorphic to the DRl-monoid of all sections as in compact support of a $T_2-$ sheaf of simple totally ordered DRl-monoids over a local Boolean space iff $A$ is quasi-regular DRl-monoid.

# Chapter 5

# MV Algebras

MV algebras (many valued algebras) were introduced by C. C. Chang in order to establish an algebraic support for many valued logic (including infinitely valued logic)such as Boolean Algebrs form an algebraic suppose for two valued logics. Chang developed the basis of MV algebras and obtained a subdirect representation for MV-algebras. Later several equivalent definitions of MV-algebras were introduced by many authors, some containing very few axioms, but we give the original definition given by C.C. Chang. Here we give below many equivalences of the definitions of MV-algebras (without proofs which may be found in []) of C.C. Chang or same in part I.

**Definition 0.1.** *An algebra* $(A, +, ., ^{-}, \cup, \cap, 0, 1)$ *of species* $(2, 2, 1, 2, 2, 0, 0)$ *is called an MV algebra if the following identities are satisfied.*

| | |
|---|---|
| $A_1$. $x+y = y+x$ | $A_1^{'}$.$x.y = y.x$ |
| $A_2$. $x+(y+z)=(x+y)+z$ | $A_2^{'}$. $x.(y.z)=(x.z)$ |
| $A_3$. $x+\overline{x} = 1$ | $A_3^{'}$. $x.\ \overline{x} = 0.$ |
| $A_4$. $x+0=x$ | $A_4^{'}$. $x.1=x$ |
| $A_5$. $x+1= 1$ | $A_5^{'}$. $x.0=0$ |
| $A_6$. $\overline{x+y} = \overline{x}.\overline{y}$ | $A_6^{'}$. $\overline{x.y} = \overline{x} + \overline{y}$ |
| $A_7$. $x= \overline{\overline{x}}$ | |
| $A_8$. $\overline{0} = 1$ | |
| $A_9$. $x\cup y = (x.\overline{y}) + y$ | $A_9^{'}$. $x\cap y = (x + \overline{y}).y$ |
| $A_{10}$. $x\cup y = y \cup x$ | $A_{10}^{'}$. $x\cap y = y \cap x$ |
| $A_{11}^{'}$. $x\cup(y \cup z) = (x \cup y) \cup z$ | $A_{11}^{'}$. $x \cap(y \cap z) = (x \cap y) \cap z$ |
| $A_{12}$. $x+(y\cap z) = (x + y) \cap (y + z)$ | $A_{12}^{'}$. $x.(y\cup z) = (x.y) \cup (x.z)$ |

**Example 0.1.** *(i) The nontrival example for MV-algebras is the propositional calculus of many valued propositions. (ii) A nontrival classical example is the unit interval [0, 1] of real numbers, with* $x\oplus y = max(0, x + y - 1)$, $\overline{x} = 1 - x$, $x \cup y = max(x, y)$, $x \cap y = min(x, y)$, $x.y =$ *usual xy of real numbers. Then* $([0, 1], \oplus, ., ^{-}, 0, \cap, \cup, 1)$ *is a MV-algebra. In fact if* $S \subseteq [0, 1]$ *with*
*(1)* $0 \in S, 1 \in S$

(2)$x,\ y\in S \Rightarrow$ *min(1,* $x+y)\in S$

*(3)* $x, y \in S \Rightarrow max(0, x + y - 1) \in S$

*(4)* $x \in S \Rightarrow 1 - x \in S$.

*Then* $(S, \oplus, .,^{-}, \bigcap, \bigcup, 0, 1)$ *in a MV-algebra.*

**Example 0.2.** *(iii) Any Boolean Algebra* $(B, \cup, \cap,^{'}, 0, 1)$ *is a MV-algebra with* $+ = \cup$, $. = \cap$, $^{-} = {}^{'}$

*(iv) Any representative DRl-monid with least element 0 and unity 1 in which* $(a-b)\cap(b-a) = 0$ *is a MV-algebra.*

Though there are several equivalent axioms satisfying for DRl-monoid, we give below

**Definition 0.2.** *An MV-algebra is an algebra* $(A, \oplus,^{\neg}, 0)$ *of species* $(2, 1, 0)$ *satisfying.*

*(i)* $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

*(ii)* $x \oplus y = y \oplus x$.

*(iii)* $x \oplus 0 = x$

*(iv)* $\neg(\neg x) = x$

*(v)* $x \oplus \neg 0 = \neg 0$

*(vi)* $\neg(\neg x \oplus y) = \neg(x \oplus y) \oplus x$.

5.1.1 Now we give below some simplest algebraic formula and which that Mv-algebras satisfy. Let $A$ be an MV-algebras and let $a, b, x, y, z$ are elements of $A$.

(1) $a \leq b$ means $a \cup b = b$ or equivalently $a \cap b = a$. Then order is a partial order.

(2) $x.y \leq x \leq x + y$, for all $x, y$ in A.

(3) Any MV-algebra is a distributive lattice with 0 as least and 1 as greastest with $\cap$ and $\cup$ as glb and lub respectively.

(4) $x.y \leq x \cap y \leq x \leq x \cup y \leq x + y$

(5) x$\leq y \Leftrightarrow \overline{x} + y = 1 \Leftrightarrow x\overline{y} = 0$

(6) $x + y = x \Leftrightarrow y \cup \overline{x} = 1 \Leftrightarrow x \cap \overline{y} = 0$

(7) $x\overline{y} + y = y\overline{x} + x$

(8) y(x+z)$\leq x + y$

(9) $(x + \overline{y}) \cup (y + \overline{x}) = 1$

(10) $x \cap y = 0 \Rightarrow x + yz = y(x + z)$

(11) $x \cup y = 1 \Rightarrow (x.x) \cup (y.y) = 1$

(12) For any $x$ in $A$ and for any non negative integer $n$, we define $x^0 = 1$, $x^{n+1} = (x^n)x$, $0.x = 0$, $(n + 1)x = (nx) + x$, with notation, $x \cup y = 1 \Rightarrow x^n \cup y^n = 1$ for each positive integer $n$.

(13) $x \cap y = 0 \Rightarrow (nx) \cap (ny) = 0$

(14) Any MV-algebra is a subdirect of totally ordered MV-algebras. This theorem is referred to as Chang Theorem.

5.1.2(i) Any MV-algebra is a DRl-monoid.

**Proof.** Let $(A, +, .,^{-}, 0, 1, \cup, \cap)$ be a MV-algebra.

For $a, b$ in $A$, with $a - b = a.\overline{b}$.

*Now* (a-b)+b= a.$\overline{b} + b = a \bigcup b \geq a$. Let $x + b \geq a \Rightarrow (x + b)\overline{b} \geq a\overline{b} \Rightarrow x \cap \overline{b} \geq a\overline{b} \Rightarrow x \geq a\overline{b}$. *Thus* a$\overline{b}$ is the least such that $x + b \geq a$.

Also $a\bar{b}+b = a\cup b$ gives the identity $(a-b)\cup 0+b = a\cup b$. Also $a+(b\cup c) = (a+b)\cup(a+c)$ follows from the fact that it is satisfied in any totally ordered MV-algebra and by Changs theorem follows that identity is valid in any MV-algebra.

$a+(b\cap c) = (a+b)\cap(a+c)$ as $A_{12}$ of the definition of MV-algebra. Thus any MV-algebra is a DRl-monoid with least element 0, greatest element 1, $a-b = a\bar{b}$. Also in every totally ordered MV-algebra, for any $a,b$ we have $(a-b)\cap(b-a)=0$ and so by Changs theorem, in any MV-algebra $(a-b)\cap(b-a)=0$. Thus any MV-algebra is a representable DRl-monoid.

5.1.2(ii) For any $x,y$ in a MV-algebra $A$, we write $d(x,y) = x\overline{y}+\overline{x}y$, $d(x,y)$ is called the symmetric difference of $x$ and $y$.

(iii) $d(x,y)=0$

$d(y,x) = x\overline{x}+y\overline{y} = 0+0 = 0$

$d(x,y) = d(y,x)$, clear from definition of $d(x,y)$

$d(x,y) = d(\overline{x},\overline{y})$, clear as $\overline{\overline{x}} = x$, $\overline{\overline{y}} = y$

$d(x,0) = x\overline{0}+0\overline{x} = x.1+0 = x$

(iv) $d(x,y)=0 \Rightarrow x=y$

$d(x,y)=0 \Rightarrow x\overline{y}+y\overline{x}=0 \Rightarrow x\overline{y}=0=y\overline{x} \Rightarrow x\le y$ and $y\le x \Rightarrow x=y$

(v) $d(x,z)\le d(x,y)+d(y,z)$, for any $x,y,z$ in $A$.

**Proof.** $\overline{z\overline{x}}+\overline{x}y+\overline{y}z = \overline{z}+x+\overline{x}y+\overline{y}z = (x+\overline{x}y)+(\overline{z}+\overline{y}z)$, by definition of $\cup$

$= y+x\overline{y}+\overline{y}+y\overline{z} = 1$, so that $z\overline{x}\le\overline{x}y+\overline{y}z$. Similarly $x\overline{z}\le x\overline{y}+y\overline{z}$ $\Rightarrow$d(x, z)= x$\overline{z}+\overline{x}z \le x\overline{y}+\overline{y}z+x\overline{y}+y\overline{z}=$ $d(x,y)+d(y,z)$

(vi) $d(x+v,y+u)\le d(x,y)+d(u,v)$

**Proof.** $d(x+v,y+u) = \overline{x+u}(y+v)+(x+u)(\overline{y+v}) \le \overline{x}y+y\overline{x}+\overline{u}v+\overline{v}u$ is to be shown. Now $(x+u+\overline{y\overline{v}})+\overline{x}y+\overline{u}v=$ (x+$\overline{x}y$) + $\overline{y\overline{v}}$ + $(u+\overline{u}v)$ = $(y+x\overline{y})+\overline{y\overline{v}}+(v+u\overline{v})$ = $x\overline{y}+u\overline{v}+(y+\overline{y\overline{v}})+v$= $x\overline{y}+u\overline{v}+v+\overline{v}+yv$ = 1 (as v+$\overline{v}$ = 1) *which implies* $(\overline{x+u})(y+v)\le\overline{x}y+\overline{u}v$. *Similarly, we obtain* $(x+u)(\overline{y+v})\le\overline{y}x+\overline{v}u$. Hence $d(x+u,y+v)\le d(x,y)+d(u,v)$

(vii) $d(xu,yv)\le d(x,y)+d(u,v)$

**Proof.** $d(xu,yv) = d(\overline{xu},\overline{yv}) = d(\overline{x}+\overline{u},\overline{y}+\overline{v}) \le d(\overline{x},\overline{y})+d(\overline{u},\overline{v}) = d(x,y)+d(u,v)$

(viii)Theorem: Any MV- algebra $A$ is a regular automatized algebra which implies $(A,A,d)$ is a OMM- space.

**Proof.** $(iii),(iv),(v)$*willgivetheresult.*

(*ix*) *Every isometry of a MV* $-$ *algebra is given by* x$\rightarrowtail d(x,a)$ for some $a$

**Proof.** As $d$ is regular, theorem 1.5.4 gives the result.

(x) Define $B = \{a\in A | x\rightarrowtail d(x,a) isanisommetry\}$. If $*$ divides $d$, then, $(B,*,0,\cap)$ is a Boolean ring.

**Proof.** Any MV-algebra is a representable DRl=monoid with least element 0 and with unity, and the distance as in MV-algebra is same as the distance in DRl-monoid. Hence by theorem 4.1.2(iv) follows that $(B,*,o,\cap)$ is a Boolean ring.

(xi) The group of isometries of $A$ is isomorphic with the group $(B,*,0)$.

(xii) If $B$ is the group of isometries as $A$, then $(B,\cup,\cap,',0,1)$ is a Boolean algebra, where $\cup,\cap,'$ are the operations in $A$.

**Proof.** we need only to show $1\in B$, i.e, $x\rightarrowtail 1*x isanisometry. Now$ $(1*x)*(1*y) = (1-x)*(1-y) =$

$((1-x)-(1-y))\cup((1-y)-(1-x))=\{(1-(1-y))-x\}\cup\{(1-(1-y)-y)\}=(y-x)\cup(x-y)=x*y$ *as* 1-(1-x)-x, *for all* x.

$\Rightarrow$ the mapping $x \rightarrowtail 1-x$ is distance preserving. Also $1*(1*x)=1-(1-x)=x$ implies the mapping $x \rightarrowtail 1*x$ is a surjection. As $(B,*,0,1)$ is a Boolean ring with unity, follow with $(B,\bigcup,\bigcap,^{'},0,1)$ is a Boolean algebra.

(xiii) In any MV-algebra, $x \rightarrowtail a*x$ is an isometry iff $a$ is a complemented element in $A$.

**Proof.** $a\in B \Leftrightarrow x \rightarrowtail a*x$ is an isometry

$a\in B \Rightarrow$a *is complemented by* x.

*Now Suppose* a$\in A$, a complemented. Let $A$ be embedded as a subdirect product of $\prod_{i\in I}A_i$, where each $A_i$ is a totally ordered MV-algebra (By Changs theorem)

Let $\sigma$ denote the mapping $x \rightarrowtail a*x$. For each $i\in I$, define $\sigma_i : A_i \rightarrow A_i$ by $\sigma_i(x_i)=a_i*x_i$, where $a_i$ is the each complement of $a$ in $\prod_{i\in I}A_i$. $a$ is complemented implies each $a_i$ is complemented in $A_i$. This implies $a_i=0_i$ or $1_i$ as $A_i$ is totally ordered (we use 0, 1 for $0_i$ *as well as* $1_i$). *Also* $\sigma_i : A_i \rightarrow A_i$ is an isometry of $A_i$ . This implies $(\sigma(x)*\sigma(y))_i=\sigma_i(x)*\sigma_i(y)=(\sigma_i*x_i)*(a_i*y_i)=x_i*y_i$.

$\Rightarrow$ $\sigma$ *is distance preserving. Now* y$\in A$, put $x=a*y$ so that $\sigma(x)=a*(a*y)=(a*0)*(a*y)=0*y=y$, so that $\sigma$ is a surjection. Hence the result.

(xiv) Any Boole algebra is the Boolean algebra but of all isometries of some Mv-algebra.

**Proof.** If $B$ is any Boolean algebra, then $(B,\cup,\cap,^{'},0,1)$ is the MV-algebra $(B,+,.,-,0,1,\cup,\cap)$, where $+=\cup$, $.=\cap$, $-=^{'}$. Hence it is the Boolean algebra of all isometries (of MV- algebra)

# Chapter 6

# Boolean like Rings

## 1 Fosters Boolean likerings

rings. Later Swaminathan further characterized Boolean-like rings and developed the structure theory besides studying geometry.

**Definition 1.1.** *A Boolean-like ring R is a commutative ring with unity of characteristics two in which* $a(1-a)b(1-b)=0$ *for all* $a, b$ *in R.*

We list out below some necessary algebra developed for Boolean-like rings.
For a Boolean-like ring R and $a$ in R

1. $a = -a$
2. $a^4 = a^2$
3. $a$ can be written uniquely as $a = n + e$, where $n$ and $e$ are nilpotent and idempotent elements of R respectively. In fact, $a = (a + a^2) + a^2$, $a^2$ is idempotent and $a + a^2$ is nilpotent as $(a + a^2)^2 = 0$.
4. The subset B of idempotent elements of R forms a sub-ring of R and B is a Boolean-ring with unity, and the set N of nilpotent elements of R forms a zero sub-ring of R ( in N, $mn = 0$ for every $m, n$).
5. N is an ideal of R and $R/N$ is isomorphic to B.
6. $R = \{0, 1, n, n + 1\}$ with $+$ and $\cdot$ defined by the table given below is a Boolean-like ring which is not a Boolean ring.
7. The only sub directly irreducible Boolean-like rings are the two element Boolean rings and the four-element Boolean-like ring $\{0, 1, n, n + 1\}$ defined in (6) above.
8. Any Boolean-like ring is a sub-direct product of a family of Boolean-like rings $\{R_i : i \in I\}$ , where each $R_i$ is either the two element Boolean ring or the four element Boolean-like ring $\{0, 1, n, n+1\}$. This four element Boolean-like ring is denoted by $N_4$.

| + | 0 | 1 | n | n+1 |
|---|---|---|---|---|
| 0 | 0 | 1 | n | n+1 |
| 1 | 1 | 0 | n+1 | n |
| n | n | n+1 | 0 | 1 |
| n+1 | n+1 | n | 1 | 0 |

| · | 0 | 1 | n | n+1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | n | n+1 |
| n | 0 | n | 0 | n |
| n+1 | 0 | n+1 | n | 1 |

9. Let $B_0$ be any Boolean ring and $N_0$ be a module over $B_0$. Put $R_0 = B_0 \oplus N_0$ (as groups). Then, define a binary multiplication " $\cdot$ " on $R_0$ such that $(R_0, +, \cdot, 0, 1)$ is a Boolean-like ring, where the set of all idempotent in $B_0$ and the set of all nilpotent in $N_0$.
   Conversely, if R is any Boolean-like ring with B as the set of all idempotent, N as the set of all nilpotent, then R is isomorphic with $B \oplus N$.

**Theorem 1.1.** *Let $R$ be any Boolean-like ring and $B$ be the set of all idempotent elements of $R$. For $a, b$ in $R$, define $a \geq b$ if there exists $x$ in $B$ such that $ax = b$.*

1. *$\geq$ is a partial ordering on $R$.*

   *Proof.* For any $a$ in R, $a = a \cdot 1$, 1 in B so that $a \geq a$.
   Let $a \geq b$ and $b \geq a$ for $a, b$ in R. There exist $x_1, x_2$ in B such that $ax_1 = b$ and $bx_2 = a$. So, $x_1x_2a = x_2b = a$ and $x_1x_2a = x_1^2x_2a = x_1a = b$. Thus $a = b$.
   Let $a \geq b$ and $b \geq c$ for $a, b, c$ in R. Then $x_1a = b$ and $x_2b = c$ for some $x_1, x_2$ in B. This implies $x_2x_1a = c$. Hence $a \geq c$. Thus $\geq$ is a partial ordering on R. □

2. *$a \geq 0$ for all $a$ in $R$.*

3. *For $a, b$ in $R$, $a \geq b$ implies $ca \geq cb$ for all $c$ in $R$.*

   *Proof.* Assume that $a \geq b$ for $a, b$ in R. Then $ax = b$ for some $x$ in B. This implies $(ca)x = cb$ for all $c$ in R. Thus $ca \geq cb$ for all $c$ in R. □

4. *For any $a$ in $R$, $a \geq a^3$.*

   *Proof.* $a^3 = a \cdot a^2$ so $a \geq a^3$ as $a^2$ is idempotent.
   For an element $x$ in R, we have $x = x_B + x_N$, where $x_B \in B$ and $x_N \in N$. This representation is unique. $x_B$ and $x_N$ are called the Boolean and nilpotent complement of $x$ respectively. □

5. *For $a, b$ in $R$, $a \geq b$ implies $a_B \geq b_B$ and $a_N \geq b_N$.*

   *Proof.* Assume that $a \geq b$ for $a, b$ in R. Then $ax = b$ for some $x \in B$. This implies $xa_B = b_B$ and $xa_N = b_n$ . So, $a_B \geq b_B$ and $a_N \geq b_N$.
   The converse of this is not true. □

6. *$1 \geq a$ if and only if $a \in B$.*

   *Proof.* Let $a \in B$. Then $1 \cdot a = a$ and $1 \in B$. So, $1 \geq a$.
   Suppose that $1 \geq a$. Then $1 \cdot x = a$ for some $x$ in B. This implies $x = a$ and hence $a \in B$. □

7. *For $a$ in $R$ and $b$ in $B$, $a \geq b$ if and only if $ab = b$.*

   *Proof.* Assume that $a \geq b$. This implies $ab = b_B = b$ as $b \in B$.
   The converse is implied by the definition of $\geq$. □

8. *For $a, b$ in $R$, we have the following:*

   *i)* *If $a \in B$ and $b \in R$, $a \geq b \implies b \in B$.*

   *ii)* *If $a \in N$ and $b \in R$ , $a \geq b \implies b \in N$.*

   *iii)* $a \in B \implies a \geq a_N(1 + a_B)$.

   *iv)* $a \geq ab$.

   *v)* $a + b + ab \geq (ab)_B$.

   *Proof.* (i),(ii), (iv) and (v) are obvious.
   To prove (iii), $a \in R \implies 1 + a_B \in B$ and $a(1 + a_N) = a_N(1 + a_B)$. So, $a \geq a_N(1 + a_B)$.
   We remark that no nonzero element of B is greater than any nonzero element of N and vice versa. □

9. *For $a, b, x$ in $R$, if $a \geq b$ and $a + x \geq b + x$, then $ax = bx$.*

   *Proof.* Assume that $a \geq b$ and $a+x \geq b+x$ for $a, b, x$ in R. Then $ab = b_B$ and $(a+x)(b+x) = (b+x)_B = b_B + x_B$. That is, $ab + ax + bx + x^2 = b_B + x_B$. This implies $b_B + ax + bx + x_B = b_B + x_B$. Hence $ax = bx$ as R is of characteristic two. □

10. *For $a, b$ in $R$, if $a \geq b \implies a + x \geq b + x$ for all $x$ in $R$, then $R = \{0\}$.*

    *Proof.* For $a$ in R, $a \geq 0 \implies a + a_N \geq a_N$. That is, $a_B \geq a_N$. So, $a_B y = a_N$ for some $y \in B$. This implies $a_B y = a_N \in B \cap N$ and so $a_N = 0$. Thus, R is a Boolean ring. This implies that for any $a$ in R, $a \geq 0$, $a + a \geq a$. That is, $0 \geq a$. Therefore, $R = \{0\}$. □

11. *In $R$, for any $a, b$ define $a \vee b = a + b + ab$. Then $(R, \vee, 0)$ is a commutative monoid on $R$, $b \geq a \implies b = a \vee x$ for some $x$.*

    *Proof.* Monoid part is straight calculation from the definition of $b \geq a$. Then $ab = a_B$. Putting $x = a+b$, we get that $a \vee x = b$. □

**Theorem 1.2.** *For $a, b$ in $R$, define $a \star b = a^2 + b^2 (= a_B + b_B)$. Then "$\star$" is Pseudo meter in R. (That is, "$\star$" satisfies $m_1, m_2, m_3$). Then $(R, B, \star)$ is a OMM-space.*

*Proof.* $a \star b = a^2 + b^2 \geq 0$; $a \star a = a^2 + a^2 = 0$. Clearly, $a \star b = b \star a$.
Let $a, b, c$ in R. Then $(a \star b) \vee (b \star c) \geq a \star c$ for all $a, b, c$. Because $a \star c = a^2 + c^2 = a^2 + b^2 + b^2 + c^2 \leq (a^2 + b^2) \vee (b^2 + c^2)$ as for idempotent elements $x, y$, where $x + y \leq x \vee y$. This can be seen by direct calculation.
$((x + y) \cdot (x \vee y)) = (x + y) \cdot (x + y + xy) = (x + y)^2 + (x + y)xy = x + y + xy + xy = x + y$. This implies $x + y \leq x \vee y$. Thus, $\star$" satisfies $m_1, m_2, m_3$. □

*i) (B, $\star$, 0) is a group and $\star$ coincides with the symmetric difference and*

*ii) (N, $\star$) is a commutative semi-group with $\star = 0$. That is, $m \star n = 0$ for $m, n \in N$.*

*iii) $\star$ is a metric operation on R if and only if R is a Boolean ring.*

*Proof.* We need to show that $(m_2)$ holds if and only if R is a Boolean ring. Let $n \in N$. Then $n \star 0 = 0 \implies n = 0$ if we assume $(m_2)$. This implies that R is a Boolean ring. □

*iv) $a \star 0 \geq a$ for all $a$ in $R$ if and only if $R$ is a Boolean ring.*

*Proof.* $a \star 0 = a^2 + 0 = a^2 = a_B \geq a \implies a_B a = a_B$. That is, $a_B + a_B a_N = a_B$ which implies $a_B a_N = 0$ for all $a$ in R. Therefore, $(1 + n)_B (1 + n) N = 0$ for all $n$. That is, $n = 0$. Hence R is a Boolean ring. □

*v) For any $a$ in $R$ the mappings $x \mapsto a + x,\ x \mapsto a \star x$ are distance preserving while the mappings $x \mapsto a \vee x,\ x \mapsto a \cdot x$ are contractions.*

*Proof.* $(a + x) \star (a + y) = (a + x)^2 + (a + y)^2 = a^2 + x^2 + a^2 + y^2 = x^2 + y^2 = x \star y$
Now $(a \star x) \star (a \star y) = (a^2 + x^2) \star (a^2 + y^2) = (a^2 + x^2)^2 + (a^2 + y^2)^2 = a^2 + x^2 + a^2 + y^2 = x^2 + y^2 = x \star y$.
Now $(a \vee x) \star (a \vee y) = (a + x + ax) \star (a + y + ay) = (a + x + ax)^2 + (a + y + ay)^2 = (x^2 + y^2)(1 + a^2) = (x \star y)(1 + a^2) \leq x \star y$.
Lastly $(ax) \star (ay) = (ax)^2 + (ay)^2 = a^2 x^2 + a^2 y^2 = a^2 (x^2 + y^2) = a^2 (x \star y)$. This implies $(ax) \star (ay) \leq x \star y$. □

*vi) In a Boolean-like ring R, every first distance triangle has firstly.*

*Proof.* $\star$ is associative because $(a \star b) \star c = (a \star b)^2 + c^2 = (a^2 + b^2)^2 + c^2 = a^2 + b^2 + c^2 = a^2 + (b^2 + c^2) = a \star (b \star c)$ as $+$ is associative.
Also for any $x$, $x \star 0 = x,\ x \star x = 0$.
Hence every first distance $\triangle(a \star b,\ b \star c,\ c \star a)$ of $\triangle(a, b, c)$ has firstly. □

*vii) R is a metric space if in every triangle $\triangle(a, b, c)$ of (distinct vertices), at least one vertex is equal to the opposite side.*

*Proof.* Let $0 \neq n \in N$. In the triangle $\triangle(0, n, 1)$ no vertex is equal to its opposite side. This implies that there are no nonzero nilpotent elements. That is, $N = \{0\}$. Hence R is a Boolean ring and so $\star$ is a metric. □

*viii) Any equilateral triangle is a subset of $b + N$ for some $b \in B$ and each side of the triangle is zero.*

*Proof.* Let $\triangle(a,b,c)$ be an equilateral triangle. This implies $a\star b = b\star c = c\star a$. So, $a^2+b^2 = b^2+c^2 = c^2+a^2 \implies a^2 = b^2 = c^2 \implies a_B = b_B = c_B$ $(= d$ say). This implies that $a = d+n_1, b = d+n_2, c = d+n_3$. Hence $\{a,b,c\} \subseteq d+N$.
Also any side (say) $a \star b = a^2 + b^2 = d^2 + d^2 = 0$. □

**Remark 1.1.** *In R, there are equilateral triangles but each side is zero, whereas non of Boolean geometry, Browrian geometry, geometry of l-groups contains equilateral triangles.*

**Theorem 1.3.** *i)* $(a,x,b)M$ *if and only if their Boolean components are so.*

*Proof.* $(a,x,b)M \Leftrightarrow (a\star x)\vee(x\star b) = a\star b \Leftrightarrow (a_B\star x_B)\vee(x_B\star b_B) = a_B\star b_B$ as for any $x, y$in R, $x\star y = x_B\star y_B$.
Note: The mapping $x \mapsto x_B$ is called Boolean projection and idempotent, distance preserving homomorphism from R to B. □

*ii) Let* $a,x,b$ *be in R. Then*

1. $a \le x \le b \implies (a,x,b)M$. *If the converse holds, then B is a Boolean ring.*

   *Proof.* For $a,x,b$ in R, let $a \le x \le b$. This implies that $a_B \le x_B \le b_B$. That is, $x_Ba_B = a_B, x_Bb_B = x_B, b_Ba_B = a_B$.
   Now $(a\star x)\vee(x\star b) = a_B + x_B + x_B + b_B + (a_B + x_B)(x_B + b_B) = a_B + b_B = a\star b$. Thus $a \le x \le b \implies (a,x,b)M$.
   For the converse let $a \in R$. Then $(0,a_B,1)M \implies 0 \le a_B \le 1 \implies 0 \le a \le 1 \in B$. This implies $a \in B$ as $1 \in B$. Thus R is a Boolean ring. □

2. $ab \le x \le a\vee b \implies (a,x,b)M$.

   *Proof.* Let $ab \le x \le a\vee b$. This implies $(ab)_B \le x_B \le (a+b+ab)_B \implies (ab)_B \le x_B \le (a\vee b)_B$.
   This implies $(a_B, x_B, b_B)M \implies (a,x,b)M$. □

*iii) We say* $a,b,c$ *are linear if* $(a,b,c)M$ *is. Note that if these points are linear, then* $ab$ *imaterially lies between the other two.*

**Theorem 1.4.** *If* $b_1,b_2,b_3$ *are linear, then any three points of* $b_1+N, b_2+N, b_3+N$ *which lying on one of them are linear.*

*Proof.* $(b_1,b_2,b_3)$ are linear implies $(b_1,b_2,b_3)M \implies (b_1\star b_2)\vee(b_2\star b_3) = b_1\star b_3$. That is, $[(b_1+n_1)\star(b_2+n_2)]\vee[(b_2+n_2)\star(b_3+n_3)] = (b_1+n_1)\star(b_3+n_3)$ for any $n_1,n_2,n_3 \in N$. i.e, any three points $b_1+n_1,\ b_2+n_2,\ b_3+n_3$ of $b_1+N,\ b_2+N,\ b_3+N$ are linear. □

iv) Any three points of $b_1+N, b_2+N$ are linear.

*Proof.* If all the three points are in the same $b_1 + N$ or $b_2 + N$, then obviously they are linear. Assume $b_1+n_1, b_1+n_2, b_2+n_3$ are our three points. Then $[(b_1+n_1)\star(b_1+n_2)]\vee[(b_1+n_2)\star(b_2+n_3)] = b_1+b_2 = (b_1+n_1)\star(b_2+n_3)$ □

v) Recall that three points $a, b, c$ are said to have a special inner point property if $(a, b, c)M$ and $(a, c, b)M$ implies $b = c$.

**Theorem 1.5.** *Three points of R fail to have special inner point property if and only if they belong to either* $b + N$ *or* $b_1 + N,\ b_2 + N$ *for some* $b,\ b_1,\ b_2 \in B$.

*Proof.* Let $x, y, z \in R$, $(x, y, z)M$ and $(x, z, y)M$ with $y \neq z$. This implies $(x_B \star y_B) \vee (y_B \star z_B) = x_B \star z_B$ and $(x_B \star z_B) \vee (z_B \star y_B) = x_B \star y_B$. That is, $(x_B + y_B) + z_B + x_B y_B + z_B x_B = x_B + z_B$ and $x_B + z_B + y_B + x_B z_B + z_B x_B + y_B x_B = x_B + y_B$. i.e., $x_B + z_B = x_B + y_B$ so that $z_B = y_B$. $z_B = y_B$ and $y_B \neq x_B$ or $x_B = y_B = z_B$. This implies $y, z$ lie on $y_B + N$ and $x$ lies on $x_B + N$ or $x, y, z$ lie on $x_B + N$.
Conversely, if $x_B = y_B = z_B$ or $y_B = z_B \neq x_B$ reversing the steps we get the result. □

vi) R has special inner point property if and only if $R = B$.

*Proof.* Let R have special inner point property. That is, for any $a, b, c$ of R we have $(a, b, c)M$ and $(a, c, b)M \implies b = c$. Now $(1, 0, n)M$ and $(1, n, 0)M$ hold for any $n \in N$. Hence $n = 0$ so that $R = B$. The converse is a result in Boolean Algebra which is true. □

vii) 1. If $\sigma$ is any isometry of R, then $\sigma$ acts as identity on B.

2. If $\sigma_1$ is the identity map of B and $\sigma_2 : N \to N$ is any involution on N, then $\sigma : R \to R$ defined by $\sigma(a + m) = a + \sigma_2(n)$ is a distance preserving surjection.

*Proof.* 1. Let $x = a + m$ and $\sigma(x) = b + n, \sigma(0) = c + p$, where $a, b, c \in B, m, n \in N$. Now $x \star 0 = \sigma(x)\star\sigma(0)$. This implies $a = (b+n)\star(c+p) = b+c$. Also, $0 = 0\star0 = \sigma(0)\star\sigma(0) = c^2+p^2 = c+0 = c$. So, $a = b$. Hence $\sigma(x) = a + x$. i.e., $a + x \mapsto a + n$.

2. Let $x = a + m$, $y = b + n$. This implies $\sigma(x) = a + \sigma_2(m)$, $\sigma(y) = b + \sigma_2(n)$. So, $\sigma(x) \star \sigma(y) = (a+\sigma_2(m))\star(b+\sigma_2(n)) = a^2+b^2 = a\star b = x\star y$. For any $c+p \in R$, $\sigma(c+\sigma_2(n)) = a+\sigma_2(\sigma_2(n)) = c+p$ so that $\sigma$ is a surjection.

□

# Chapter 7

# Newmann Algebras

**7.1.1:** A Newman Algebra is a system $N = (N, +, ., 0, 1)$ of species $(2, 2, 1, 0, 0)$ which satisfies the following

$N_1$: $a(b + a) = a.b + a.c$

$N_2$: $(a + b).c = a.c + b.c$

$N_3$: $a.1 = a.$

$N_3$: $a.1 = q.$

$N_4$: $aa' = 0, a = a' = 1$, for all $a, b, c$ in $N$.

Note that none of idempotency, commutativity or associativity for either $'A'$ or for $(.)$ is assumed. We have the following consequences of the definition we give proofs for many.

**7.1.2(i)** $a + a = a$ for all $a$ in $N$.

*Proof.* $a.a = a.a + a.a' = a.(a + a') = a$ □

ii) $(a')' = a.$

*Proof.*

$$\begin{aligned}(a')' = 0 + (a')' &= a'(a')' + (a')'(a')' \\ &= (a' + (a')')(a')' = 1.(a')' = (a + a')(a')' \\ &= a(a')' + 0 = 0 + a(a')' = aa' + a(a')' \\ &= a(a' + (a')') = a.1 = a.\end{aligned}$$

□

iii). $a'a = 0$ and $a' + a = a, a.0 = 0.a = 0.$

*Proof.* $a'a = a'.(a')' = 0$

$0 = a.a' = a(a' + 0) = aa' + a0 = 0 + a.0 = a.0$

Also, $0 = (a')(a')' = a'a = (a' + 0)a = 0'a + 0.a = a'(a')' + 0.a - 0 + 0.a = 0.a.$ Also, $a' + a = a' + (a')' = 1.$ □

3 $1a = a$

*Proof.* $1a = (a + a')a = a.a + a'a = a + 0 = a.$. Note that if $1 = 0$, then $a = a.0 = a.1 = a$ for all $a$ in $N$ which maps $N = 0$. So we assume that $0 \neq 1$. □

**Definition 0.1.** *We define* $1+1 = 2$ *and all elements* $x2$ *are called even elements. Note that* $2+2 = 2.1+2.1 = 2(1+1) = 2.2 = 2.$. *Now we have* $x$ *is an even iff* $x + x = x$.
*Obviously,* $y.2 + y.2 = y.(2 + 2) = y.2$ *and conversely if* $x = x + x$ *then,* $x = x.1 + x.1 = x.(1 + 1) = x.2$ *which is an even.*

**Theorem 0.1.** *Any multiple* $xt$ *or* $ut$ *of an even element is even.*

*Proof.* For if $x = x + x$, then, $xt = (x + x)t = xt + xt$ and $ux = u(x + x) = ux + ux$ so that$xt$ and $ux$ are even. □

*Viii).* The set $B$ of all even elements of $N$ form a Boolean Algebra.

*Proof.* By $(vii)$ follows that $B$ is closed under $+$ and $.$. Also, $0 \in B, 2 \in B$. We claim $(B, +, .,', 0, 2)$ is a Boolean Algebra. □

We quote here a Theorem from Birkhoff(remark 34).

**Theorem 0.2.** *The following postulates on asystem with two binary operations* $\vee, \wedge$, *and an element* 1 *imply that it is a distributive lattice.*

$D_1$ $a \wedge a = a$, *for all* $ainA$

$D_2$ $a \vee 1 = 1 \vee a = a$, *for all* $a \in A$.

for $a$ even $a'2$ is the complement of $a$ in $B4$. Because $a + a'2 = a_2 + a'2 = (a + a')2 = 1.2 = 2$, and $a.(a'2) = a(a' + a') = aa' + aa' = 0$. So, by Birkhoff's theorem follows that $(B, +,', 0, 1, 2)$ is a Boolean Algebra (where $(x2)' = (x'2)$, also $(B, +,', 0, 1, 2)$ is a distributive and complemented lattice.
( In fact, for, $a$) even $a+a'2 = a2+a'2 = (a+a')2 = 1.2$ and $a(a'2) = a(a'+a') = aa'+aa' = 0$ **a) Definition**: All left multiples of $2'$ are called odd elements. now we observe that $x$ is odd iff $x + x = 0$.

*Proof.* $2' + 2' = 2'.1 + 2'.1 = 2'.2 = 0$ □

If $x$ is odd, then $x = y2'$ for some $y$ in $N$ and $4x = x = y2' + y2' = y(2' + 2') = y.0 = 0$.
On the other hand, if $x + x = 0$, then, $4x = x.1 = x.(2' + 2) = x2' + x2 = x2'$ since $x2 = x + x = 0$ so that $x$ is odd.
() $+$ is commutative in $N$. Let $a, b \in N$ and let $x = a + b$ and $y = (b + a)'$
$a + b = (a + b)' = (a + b)(x + (b + a)) = (a = b)x + (a + b)(b + a)$.

Now, $0 = (b+a)(b+a)' = (b+a)y = b.y + a.y$. Now we observed that $\alpha + \beta \implies \alpha = \beta$ for $\alpha, \beta$ in $N$.

$$\begin{aligned} \alpha &= \alpha.1 \\ &= \alpha(\beta' + \beta) \\ &= (\alpha\beta' + \beta\beta') + \alpha\beta \\ &= 0 + \alpha\beta = \alpha\beta. \end{aligned}$$

Similarly,

$$\begin{aligned} \beta = 1.\beta & \\ &= (\alpha' + \alpha)\beta \\ &= \alpha'\beta + \alpha\beta \\ &= (\alpha'\alpha + \alpha'\beta) + \alpha\beta \\ &= \alpha'(\alpha + \beta) + \alpha\beta \\ &= 0 + \alpha\beta \\ &= \alpha\beta \end{aligned}$$

Hence, $\alpha = \beta$. Now, $0 = by + ay \implies by = ay \implies ay + by = 0 \implies (a+b)y = 0$. So, $(a+b) = (a+b)(b+a)$. Similarly $b + a = (a+b)(b+a)$. Hence, $a + b = b + a$.

() **Associativity in $N$.**

We divide the proof in to 5 steps and in each step

let $x, y$ stand respectively for $a, b, c$ in $N$.

**Step 1:** $1 + (1 + a) = (1 + 1)a$.

*Proof.*

$$\begin{aligned} ax &= a1 + (a+b)] = a.1 + a(a+b) \\ &= a.1 + (a.a + a.b) = a1 + (a.1 + ab) \\ &= a.1 + a(1+b) = a[1 + (1+b)] \\ &= a[(1+1) + b] \text{by step 1} \\ &= a(1+1) + ab = (a+a) + ab \\ &= (a + aa) + ab = a(1+a) + ab \\ &= a[(1+a) + b] = ay \end{aligned}$$

□

$$\begin{aligned} a'x &= a'[1+(a+b)] = a'1 + a'(a+b) \\ &= a'1 + (ab + a'b) = a' + (0 + a'b) \\ &= a' + a'b \end{aligned}$$

$$\begin{aligned} a'y &= (a' + (1+a) + b) + a'(1+a) + a'b \\ &= (a' + a'a) + a'b = (a' + b) + a'b \\ &= a' + a'b. \end{aligned}$$

$\implies a'x = a'y.$

**Prop 3** $a + (b + c) = (a + b) + c$

*Proof.*

$$\begin{aligned} ax &= a.[a + (b+c)'] = a.a + a(b+c) \\ &= a + a(b+a) = a(1 + (b+c)) \\ &= a[(1+b)+c] \text{ by step 2} \\ &= a(1+b) + a.c = (a + ab) + ac \\ &= (aa + ab) + ac = a((a+b)+c) \\ &= ay \end{aligned}$$

$a'x = a'(a + (b + c)) = a'a + a'(b + c) = 0 + a'(b + c) = a'b + a'c.$
$a'y = a'[(a + b) + c] = a'(a + b) + a'c = (a'a + a'b) + a'c = (a + a'b) + a'c = a'b + a'c \implies a'y = a'x.$

Now, $x = (a' + a)x = a'x + ay = a'x + ay = (a' + a)y = y.$ □

() Any Newman algebra is the direct product of a Boolean algebra and a Boolean Naring with unity. (Boolean Naring means a ring with a not necessarly associative multiplication in which $a^2 = a$.).

*Proof.* Let $B$ be the set of all even elements and let $B'$ be the set of all odd elements. we have already observed that $(B, +, ', 0, 1, 2)$ is a Boolean Algebra. Also $B'$ is closed under + and . ( Can be seen in a view of associativity of + in $N$, commutativity of $'+'$ and $'.'$). □

Also the distributive laws are valid in $N$ and so in $B'$. Hence, $(B', +, ', 0, 2')$ in a Boolean Naring with unity (as $a^2 = a$ is always ...)
Also any $x$ in $N$ can be written separately as $x = x_2 + x_2' \in B \times B'$. The mapping $x \rightarrowtail (x_2, x_2')$ from $N \to B \times B'$ is an isomorphism as Newmann algebras as $B, B'$ are subnewmann algebras of $N$( with 2 as unity in $B$, 2' as unity in $B'$).

**(xiii)** Here after we write any element $n$ in $N$ as $n = b + a$ where $b \in B, a \in B'$

(1) Every element of $B'$ have a unique complement with respective 0 and $z'$ in $B'$.

*Proof.* Let $a \in B'$ put $e^* = e + z'$. Now, $e(e + z') = e + ez' = e + e = 0$ □

(2) The complement of any $x$ element $n$ in $N$ (where $n = b + e, b \in B, e \in B'$) is $n' = b^- + e^x$ where $b^-$ in the complement of $b$ in $B$ with respect to 0 and 2. and $e^*$ in the complement of $e$ in $B'$ with respect to 0 and $2'$.

*Proof.* $(b + e)(b^- + e^*) = (b + b^-) + (e + e^x) = 2 + 2' = 1$
$(b + e)(b^- + e^*) = bb^- + ee^* = 0.$ □

(3) If $n_1 = b_1 + e_1, n_2 = b_2 + e_2$ where $b_1, b_2 \in B, a_1, a_2 \in B'$ then, $n_1 n_2' + n_1' n_2 = (b_1 b_2^- + b_1^- b_2) + (b_1^- + e_1^*)(b_2 + e_2) = (b_1 b_2^- + b_1^- b_2) + e_1 + e_2$, where $b^-$ denotes the complement of $b$ in $B$ and $e^*$ denotes the complement of $e$ in $B'$.

*Proof.*

$$\begin{aligned} n_1 n_2' + n_1' n_2 & \\ &= (b_1 + e_1)(b_2 + e_2)' + (b_1 + e_1)'(b_2 + e_2) \\ &= (b_1 + a_1)(b_2^- + e_2^*) + (b_1^- + e_1^*)(b_2 + e_1^* e^2) \\ &= (b_1 b_2^- + b_1 b_2) + e_1 + e_2 \end{aligned}$$

□

(xiii)(4): For $a, b \in N$, we define $a \geq b$ iff $ab = b$. We observed that $bn_1 > b_1 + a_1, n_2 = b_2 + a_2$, then $n_1 > n_2$ iff $b_1 \geq b_2$ and $a_1 \geq a_2$.

*Proof.* $n_1 n_2 = (b_1 + a_1)(b_2 + a_2) = (b_1 b_2 + a_1 a_2)$ so $n_1 n_2 = n_2$ iff $(b_1 b_2 + a_1 a_2) = (b_2 + a_2) \iff b_1 b_2, e_1 \geq e_2$. □

(5) If $b_1 b_2$, then, for any $a \in B', b_1 + e \geq b_2 + e$

(6) $\geq$ is reflexive and antisymmetric but necessarly transitive.

(xiv): The operation $'*'$ defined by $n_1 * n_2 = n_1 n_2' + y_1' y_2$ for $n_1, n_2$ in $N$ in a metric operation.

*Proof.* $n_1 * n_2 \geq 0$ for all $n_1, n_2$ in $N$.
$n_1 * n_2 = n_1 n_1^{'} + n_1^{'} n_1 = 0$
$n_1 * n_2 = n_2 * n_1$ □

as + and . are commutative).

If $n_1 * n_2 = 0$, then, $b_1 \circ b_2 + e_1 + e_2 = 0$ (where $b_1 \circ b_2 = 0, e_1 + e_2 = 0 \implies b_1 = b_2, e_1 = e_2$).

**Ole inequality**: let $n_3 = b_3 + e_3$. Now, $(n_1 * n_2) + (n_2 * n_3) = (b_1 \circ b_2 + e_1 + e_2) * (b_2 \circ b_3 + e_2 + e_3) \geq b_1 \circ b_3 + e_1 + e_3 = n_1 * n_3$. Thus, $'*'$ satisfies $(m_1), (m_2), (m_3), (m_4)$.

(3) If one vertex of triangle is equal to its opposite side, then, the other vertices are also equal to the opposite side respectively. ie., the triangle has fixty.

*Proof.* Let $n_1 * n_2 = n_3$ for the $\triangle(n_1, n_2, n_3) \implies n_1 * n_3 = n_1 * (n_1 * n_2) = n_2$ by associativity of $*$. Similarly $n_2 * n_3 = n_1$. □

(4): Any first distant triangle has fixty and conversely.

*Proof.* Obvious $'*'$ is a group operation. Converse means every triangle with fixty is a first distance triangle of itself. □

(5): For $n_1 = b_1 + e_1, n_2 = b_2 + e_2, n_3 = b_3 + e_3, (n_1, n_2, n_3)M$ iff $(b_1, b_2, b_3)M$.

*Proof.* Straight observation. □

(6)$*$ is regular metric operation.

*Proof.* $n * 0 = n$. □

(7) Any isometry $\sigma : N \to N$ is given by $\sigma(x) = n * x$ for some $n \in N$.
(8): For each $a$ in $N$, the mapping $x \rightarrowtail a * x$ is an isometry.

*Proof.* Let $a = b + e, x = b_1 + e_1, y = b_2 + e_2$ Now, $(a * x) * (a * y) = (b \circ b_1, e + a_1) * (b \circ b_2, e + e_2) = (b \circ b_1 \circ b \circ b_2, e + e_1 + e + e_2) = (b_1 \circ b_2, e_1 + e_2) = (b_1 + e_1) * (b_2 + e_2) \Rightarrow x \mapsto a * x$ is a bijection. □

**Remark 0.1.** *We see that all isometries of N are the group translations $x \mapsto n*x$. Thus the group of isometries of N are isomorphic with the group $(N, *, 0)$.*

If $\sigma : N \longrightarrow N$ is an isometry given by $\sigma(x) = a * x$ with $a = b + e$ then there exists an isometry $\tau : B \longrightarrow B$ and a translation $T : B' \longrightarrow B'$ such that $\sigma(c + e) = \tau(c) + \tau(e)$ for any $x = c + e$

*Proof.* B,$B'$ are also autometrized spaces with $\circ$ as distance function in B and $+$ in $B'$ respectively. For any $s \in B$, define $\tau : B \longrightarrow B$ by $\tau(s) = b \circ s$ and for any $t \in B', T : B' \longrightarrow B'$ by $T(t) = t + e$. Then $\tau$ and $T$ are isometries of B and $B'$ respectively. Also $\sigma(c+m)$ for any $c+m \in N$ is given by $\sigma(c+m) = (c+m) * (b+e) = a * b + m + e = \tau(a) + \tau(m)$ □

Converse : If $\tau : B \longrightarrow B$ is an isometry and $T : B' \longrightarrow B'$ is a translation given by $\tau(s) = b \circ s, T(t) = t + m$ then $\sigma$ defined by $\sigma(a + m) = (b + n) * (e + m) = T(c) + T(m), \sigma(x) = a * x$ where $a = b + m, x = c + n$

# Chapter 8

# Griss Algebras

## 1 Introduction

Gris Algebras form an algebraic formulation of Griss negation less intutistic logic just as Boolean Algebras and MV-algebras form algebraic formulations for propostional calculus(Two valued logics) many valued logics respectively. This algebraic formulation is due to Iseki and Imai. We first introduce the definition of Griss algebras and also a genralizaion of Griss algebras called Generalised Griss algebrass due to Prabhakara Rao[ ]

**Definition 1.1.** *A system* $(G, \vee, -, 0)$ *where* $\vee, -$ *are binary operations on* $G$ *and* $0$ *is a nullary operation satisfying*

$G_1$ $(G, \vee)$ *is a join semi lattic with 0 as least*

$G_2$ $(x \vee y) - (x \vee z) \leq y \vee z$ *for any* $x, y, z \in G$

$G_3$ $(x - z) \leq (x - y) \vee (y - z)$ *for all* $x, y, z \in G$

$G_4$ $x - y = 0 \Leftrightarrow x \leq y$

*is called a Griss algebra. (Here* $\leq$ *is the ordering induced by the join operation* $\vee$*)*

**Definition 1.2.** *A system* $(G, \vee, -, 0)$ *satisfying* $G_1, G_3, G_4$ *and* $G_2$ $(x \vee y) - x \leq y \vee (y - 0)$ *is called a generalised Griss algebra.*

Note that every Griss algebra is a generalized Griss algebrs since $(x \vee y) - x = (x) - (x \vee 0) \leq y \vee 0 = y \leq y \vee (y - 0)$ The following is an example of a generalized Griss algebra which is not a Griss algebra.

**Example 1.1.** *Let* $G = \{0, a, b, 1\}$ *be four distinct elements and define* $\vee, -$ *by the tables given below Here*$(a \vee b) - (a \vee 0) = 1 - a = 1$*but* $1 \leq b = b \vee 0$

**Remark 1.1.**

1. *If* $B = (B, \vee, \wedge, ', 0, 1)$ *is a Boolean algebra then* $(B, \vee, 0, -)$ *where* $a - b = a \wedge b'$ *is a generalized Griss algebra.*
2. *If* $(A, \vee, \wedge, -0)$ *is a Browerian algebra then* $(A, \vee, -, 0)$ *is a generalized Griss algebra.*

| $\vee$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 0 | a | b | 1 |
| a | a | a | 1 | 1 |
| b | b | 1 | b | 1 |
| 1 | 1 | 1 | 1 | 1 |

| $-$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | b | 0 | b | 0 |
| b | a | a | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 |

3. *If* $(A, +, -, 0)$ *is a Semi Browerian algebra then* $(A, +, -, 0)$ *is a generalized Griss algebra.*

**Note.** There are generalized Griss algebras which are not Semi Browerian algebras.

**Definition 1.3.** *Let* $(G, \vee, -, 0)$ *be a generalized Griss algebra. We write* $a * b = (a - b) \vee (b - a)$. *We call* $a * b$ *the symmetric differnce of a and b.*

**Theorem 1.1.** *Every generalied Griss algebra* $(G, \vee, -, 0)$ *is an autometrized join semi lattice with symmetric diffference as distance function.*

*Proof.* Since 0 is the least element of G, $a * b \geq 0$ for any a, b. Also $a * a = (a - a) \vee (a - a) = 0$ (by$G_4$).Let $a * b = 0 \Rightarrow (a - b) \vee (b - a) = 0 \Rightarrow a - b = 0 = b - a \Rightarrow a \leq b, b \leq a \Rightarrow a = b$ (by$G_4$).Also $a * b = b * a$ for all a,b.Now $a * c = (a - c) \vee (c - a) \leq (a - b) \vee (b - c) \vee (c - b) \vee (b - a) = (a * b) \vee (b * c)$.$Hence$($m_1$), ($m_2$), ($m_3$) and ($m_4$) are all satisfied. Thus $(G, G, *)$ is a metric OMM-space. □

**Note**Semi Brouwerian algebras, Brouwerian algebras and Boolean Algebras are regular autometrized algebras where as generalized Griss algebras are not necessarily regular.
In fact we have the following

**Theorem 1.2.** *A generalized Griss algebra* $(G, \vee, -, 0)$ *is a Semi Brouwerian algebra* $\Leftrightarrow$ $(G, \vee, -, 0)$ *is a regular autometrized algebra.*

*Proof.* Suppose G is regular. We need only to show that $a*(a*b) \leq b$..Now $a*(a\vee b) = [a-(a\vee b)]\vee[(a\vee b)-a] = (a \vee b) - a$. Since $a \leq a \vee b \leq b \vee (b - a)$ $b \vee (b * 0) = b \vee b = b$.Hence G is a semi Browerian algebra. The other part is obvious as any Semi Brouwerian algebra is a generalized Griss algebra. □

**Theorem 1.3.** *Let* $(G, \vee, -, 0)$ *be a generalized Griss algebra. Then* $(G, *, 0)$ *is a Boolean geometry* $\Leftrightarrow$ *for each a, the mapping* $x \longmapsto a * x$ *is an isometry.*

*Proof.* Suppose $x \longmapsto a * x$ is an isometry for any a. Now $x * 0 = (0 * x) * (0 * a) = (x * 0) * 0$.Since the mapping $x \longmapsto a * x$ is an isometry implies that it is an injection and $x * 0 = x$ for any x so that $*$ is regular. So $(G, \vee, -, 0)$ is a Semi Brouwerian algebra.Also $*$ is cancellative as the mappings $x \longmapsto a * x$ are surjections. This implies G is free of isoceles triangles.Now the proof follows from theorem 2.4.2
The other part is a result in Boolean geometry which is true. □

We use the following definition of a Griss algebra.

**Definition 1.4.** *A system* $(G, \vee, -, 0)$ *is a Griss algebra* $\Leftrightarrow G_1, G_3, G_4$ *and* $G_5 : (x \vee y) - z = (x - z) \vee (y - z) \forall x, y, z \in G$ *hold in* $G$

**Theorem 1.4.** *Every Griss algebra is a generalized Griss algebra.*

*Proof.* It is enough to show that $G_2$ is satisfied. Now $(x \vee y) - x = (x - x) \vee (y - x) = 0 \vee (y - x) = y - x \leq (y - 0) \vee (0 - x)$ (by $G_3$)$\leq (y - 0) \vee 0 = y - 0 \leq y \vee (y - 0)$. Hence the theorem. □

**Lemma 1.1.** *Let* $(G, \vee, -, 0)$ *be a Griss algebra and let x,y,z,....,a,b,c.... denote the elements of* $G$ $x \leq y \Rightarrow x - z \leq y - z$

*Proof.* $y - z = z - y(x \vee y) - z = (x - z) \vee (y - z) \Rightarrow x - z \leq y - z$ □

**Lemma 1.2.** $x \leq y \Rightarrow z - y \leq z - x$

*Proof.* By $G_3$, $z - y \leq (z - x) \vee (x - y) = (z - x) \vee 0 = z - x$ □

**Lemma 1.3.** $(x \vee y) - (x \vee z) \leq y - z$

*Proof.* $(x \vee y) - (x \vee z) = (x - (x) \vee (y - (x \vee z)) = 0 \vee (y - ((x \vee z)) \leq y - z$ (by (iv)) □

**Theorem 1.5.** *Every Griss algebra is autometrized by* $*$ *where* $a * b = (a - b) \vee (b - a)$

*As every Griss algebra is a generalized Griss algebra follows the result and also G is regular* $\Leftrightarrow$ *G is a Semi Brouwerian algebra.*

**Lemma 1.4.** $(a \vee b) * (c \vee d) \leq (a * c) \vee (b * d)$

*Proof.* $(a \vee b) * (c \vee d) = [(a \vee b) - (c \vee d)] \vee [(c \vee d) - (a \vee b)] = (a - (c \vee d) \vee (b - (c \vee d) \vee (c - (a \vee b)) \vee (d - (a \vee b)) \leq ((a - c) \vee (b - d) \vee (c - a) \vee (d - b) = (a * c) \vee (b * d)$ □

***Notation*** *In a Griss algebra, for any a,b we write* $a \cdot b = (a \vee b) - (a * b)$

**Theorem 1.6.** *In a Griss algebra, the following are equivalent.*

1. $a - (b - a) = a$ *for all* $a, b \in G$
2. $(G, *, \cdot, 0)$ *is a Boolean ring.*
3. $a \cdot b$ *is the glb of a and b.*
4. $a \vee (b \cdot c) = (a \vee b) \cdot (a \vee c)$
5. $(G, *)$ *is a loop.*
6. $*$ *is a group operation.*
7. $*$ *is associative.*

**Example 1.2.** *Let G be the Griss algebra* $\{o, a, b, 1\}$ *of four distinct elements with 0 defined as*

| $\vee$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 0 | a | b | 1 |
| a | a | a | 1 | 1 |
| b | b | 1 | b | 1 |
| 1 | 1 | 1 | 1 | 1 |

| $-$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | b | 0 | b | 0 |
| b | a | a | 0 | 0 |
| 1 | 1 | a | b | 0 |

Let $(G, \theta)$ be the Kleen's four group.Then $\theta$ is a metric group operation on G but $a\theta 0 = a$ where as $a * 0 = b$. This example shows that there is a metric group operation which is not the symmetric difference.

**Theorem 1.7.** *A Griss algebra which is a chain is free of triangles with fixity.*

*Proof.* Let $\triangle\ (a, b, c)$ be a triangle with fixity. We may assume $a < b < c$. We get $a * b = c, b * c = a, c * a = b$ that $b - a = c, c - b = a$ and $c = a - b$ . Now $a = b \vee c = (b - a) \vee (c - a) = (b \vee c) - a) = c - a = b$ which condradicts that $\triangle\ (a, b, c)$ is a triangle. Hence the result. □

**Definition 1.5.** *For any a,b,c in a generalised Griss algebra $(G, \vee, -, 0)$ we (recall as usual) say b lies between a and c as semi lattice $(a\cdot) \Leftrightarrow (a \cdot b) \vee (a \cdot c) = b = (a \vee b) \cdot (b \vee c)$. In symbols we denote this by $(a, b, c)S$. This is called semi lattice betweeness. We always have metric betweeness which says $(a, b, c)M \Leftrightarrow (a * b) \vee (b * c) = a * c$*

**Theorem 1.8.** *Let $(G, \vee, -, 0)$ be a generalized Griss algebra. Then $(G, *, \cdot)$ is a Boolean geometry $\Leftrightarrow$ semilattice betweenness coincides with metric betweenness.*

*Proof.* Supose in G both concepts of betweenness coincide. For any $a \in G$, we have $(a, a, 0)M$.So by hypothesis, we have $(a, a, 0)S$ so that $(a \vee a) \cdot (a \vee 0) = a$ which implies $a^2 = a$ and hence $a * 0 = a$ for every $a \in G$ so that G is a semi Brouwerian algebra. Hence follows that G is a Boolean geometry. The other part is a result in Boolean geometry which is true. □

**Theorem 1.9.** *In a generalized Griss algebra metric betweenness has transitivity $t_2$*

*Proof.* follows as G is also a OMM-space. □

**Theorem 1.10.** *In a generalized Griss algebra, it metric betweenness has $t_1$ the G is of free of isosceles triangles.*

*Proof.* Let $\Delta(a, b, c)$ be a triangle with $a * b = a * c$. Since $v * c \leq (b * a) \vee (a * c) = a * b$ we have $(a * b) \vee (b * c) = a * b = a * c$. Hence (a,b,c)M. Since metric betweenness has $t_1, (a, b, c)M$ and $(a, c, b)M \Rightarrow (a, b, c)M$which imply $b = c$ so that the triangle degenerates which is a contradiction. Hence the result. □

**Definition 1.6.** *We say a generalized Griss algebra$(G, \vee, -, 0)$ is symmetric $\Leftrightarrow (a \vee b) * (a \cdot b) = a * b$ for any $a, b \in G$*

**Theorem 1.11.** *A generalized Griss algebra is symmetric $\Leftrightarrow$ it is a semi Brouwerain algebra.*

*Proof.* Suppose G is symmetric. Since $(a \vee b) * (a \cdot b) = a * b$ by taking $a = b$, we get $a * (a \cdot a) = 0$ which implies $a \cdot a = a$.Hence G is a semi Brouwerian algebra.Conversly suppose $(G, \vee, -, 0)$ is a semi Brouwerian algbera. First we observe that $(a \vee b) * (a \cdot b) = (a) * ((a \vee b) - (a * b)) = ((a \vee b) - [(a \vee b) - (a * b)]) \vee ([(a \vee b) - (a * b)] - (a \vee b)) = (a \vee b) - [(a) - (a \vee b)]$. since $(a \vee b) - (a * b) \leq a \vee b$ Hence $[(a \vee b) * (a \cdot b)] - (a * b) = (a \vee b) - [(a \vee b) - (a * b)] - (a * b) = (a \vee b) - [(a \vee b) - (a * b) = (a \vee b) - [(a \vee b) - (a * b)] \vee (a * b) = (a \vee b) - [(a \vee b) * (a \vee b)] = 0$.Hence $a \vee b) * (a \cdot b) \leq a * b$. Since $(a \vee b) - (a * b) = ab = a - (a * b) \leq a$, we have $b - a \leq b - [(a \vee b) - (a * b)]$. Since $(a \vee b) - (a * b) = ab = b - (a * b) \leq b$. We have $a - b \leq a - [(a \vee b) - (a * b)]$.Hence $a * b = (a - b) \vee (b - a) \leq a - [(a \vee b) - (aastb)] \vee b - [(a \vee b) - ((a * b] = [(a \vee b) - (a * b)] = [(a \vee b) - (ab)]$. Hence $(a \vee b) * (ab) = a * b$ so that G is symmetric. □

**Theorem 1.12.** *Let G be a Griss algebra in which $a * 0 \leq a$ for all $a \in G$. Then G is a Boolean geometry $\Leftrightarrow ac \leq b \leq a \vee c \Rightarrow (a, b, c)L$*

*Proof.* Suppose $ac \leq b \leq a \vee c \Rightarrow (a, b, c)L$ Now $a \cdot a = (a \vee 0) - (a * a) = a - 0 = a * 0 \leq a$.Hence $a \cdot a \leq a \leq a \vee a$ for all $a \in G$.Hence $(a, 0, a)L$(by hypothesis).Hdence $(a \vee a) \cdot (a \vee a) = a$ so that $a = a \cdot a = a * 0$ for all $a \in G$.Hence G is a semi Brouwerian algebra. Hence G is a Boolean geometry. For the other part, in a Boolean geometry it is clear. □

# Chapter 9

# Boolean metric spaces

## 1 Definition and examples

**Definition 1.1.** *A Boolean metric space is a triple $(X, B, d)$ where $X$ is a non empty set, $B$ is a Boolean algebra $(B, \vee, \wedge, ', 0, 1)$ and $d : X \times X \longrightarrow B$ is a mapping such that*

1. *$d(x, y) = 0 \Leftrightarrow x = y$*
2. *$d(x, z) \leq d(x, y) \vee d(z, y)$ for all $x, y, z \in X$.*

***Note.*** *If $(S, B, d)$ is a Boolean metric space, we also say that $(S, B, d)$ is a B-metric space, d Boolean distance function for S.*
*Taking $y = x$ in (2) above we get $d(x, z) \leq d(z, x)$. Hence by interchaning x,z we get $d(z, x) \leq d(x, z)$ so that $d(x, z) = d(z, x)$ for all x,z. Thus Boolean metric spaces satisfy $(m_1)$,$(m_2)$,$(m_3)$ and $(m_4)$. Boolean metric spaces $(X, B, d)$ form a subclass of merix OMM spaces as the ordered monoid $(B, \vee, 0)$ and d as a distance function.*

*We now give some examples*

**Example 1.1.** *Let S be a nonempty set and $B = \{0, 1\}$ be the two element Boolean Algebra. Defined by*

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases} \quad (1.0.1)$$

**Example 1.2.** *Let $B = (B, +, \cdot, ', 0, 1)$ be a Boolean Algebra, S=B,d(a,b)=$ab' + a'b$*

**Example 1.3.** *Let $R = (R, +, \cdot, 0, 1)$ be a p-ring with unity where p is prime. Let $S = R$, $B$ the set of all idempotent and $d(a, b) = (a - b)^{p-1}$*

**Definition 1.2.** *Let $S$ be a B-metric space (S,B,d)where $B = (B, +, \cdot, ', 0, 1)$ and $x, y, z \in S$.We say x,y,z are collinear $\Leftrightarrow \delta(x, y, z) = d(x, y)d(y, z)d(z, x) = 0$.A subset $T$ of $S$ such that every points of $T$ are ollinear is called a linear subset of $S$*

*Now we have the following*

**Theorem 1.1.** *Let $(S, B, d)$ be a Boolean metric space. Then $S$ is linear $\Leftrightarrow$ $S$ can be isometrically (= congruently) embedded in $(B, B, d)$*

***Note.*** *In $(B, B, d)$, $d$ is the symmetric difference. If $(S, B, d_1), (S, B, d_2)$ are two B-metric spaces, we say a mapping $\sigma : S \longrightarrow X$ is called an isometry $\Leftrightarrow$ $\sigma$ is a bijection and $d_2(\sigma(x), \sigma(y)) = d(x, y)$. This reads as $\sigma(x) * \sigma(y) = x * y$ if $*$ is used in place of $d$ with inset notation.*

*Proof.* of the theorem. Assume S is linear.Fix some $x \in S$, define $\sigma : S \longrightarrow B$ by $\sigma(y) = d(x, y)$ for some $y \in S$. We first claim that $\sigma$ is distance preserving. We use $*$ for the distance $d_2$ in B and $*$ is the symmetric difference. Let $y, z \in S$. We need to show that $\sigma(x) * \sigma(y) = d(x, y)$ where we are using d for $d_1$ in $(S, B, d_1)$ Let $a = d(x, y), b = d(x, z), c = d(y, z)$. Since S is linear we have $abc = 0$.
Now $d(x, y) * d(x, z) = a * b = ab' + a'b$
$= (ab' + a'b)(c + c') = ab'c + ab'c' + a'bc + a'bc'$
$= (ab'c + a'b)c + ab'c' + a'bc' = (ab'c + a'b)c + 0 + 0 = c + 0 + 0 = c = d(y, z)$
as $abc = 0 \Rightarrow ab \leq c', ca \leq b', cb \leq a'$ and the triangle inequality $a \leq b + c, b \leq c + a, c \leq a + b \Rightarrow ab'c' = 0 = bc'a' = ca'b'$. Thus $\sigma$ is distance preserving and consequently $\sigma$ is an injection.
For the converse part, assume that S can be isometrically embedded in B. Here we may assume without loss of generality , $S \subseteq B$ and $d(x, y) = x * y$ for $x, y \in S$. Now it is enough to show that $(x * y)(y * z)(z * x) = 0$ for any $x, y, z \in S$. A straight calculation using the distributivity of $*$ over $\cdot$,commutativity and associativity of $*$and using $a * 0 = a, a * a = 0$ we get $(x * y)(y * z)(z * x) = 0$

□

# Betweenness

**Definition 1.3.** *Let $x, y, z \in (S, B, d)$ a Boolean metric space. We say $y$ lies between $x$ and $z \Leftrightarrow x, y, z$ are collinear. In symbols we denote this by $(x, y, z)B$.*

*For $x, z \in S$, we write $[x, z] = \{y \in S | (x, y, z)B\}$*

**Definition 1.4.** *Let $x \in S$. Then $y, z \in S$ are said to be x-compatible $\Leftrightarrow$ there exists an element $u \in S$ such that $y, z \in [x, u]$*

**Theorem 1.2.** *If $x \in S$. Then every x-compatible subset of $S$ is a linear set.*

*Proof.* Let T be a x-compatible subset of S. Let $y, z \in [x, u]$. Then it follows that $d(x, z) = d(x, y)d'(x, z) + d'(x, y)d(x, z)$.Consequently the mapping $\sigma(y) = d(x, y)$ is a congruent embedding and so the theorem follows from the theorem 9.1.1(iii).

□

**Theorem 1.3.** *1.* $(x, y, z)B \Leftrightarrow (z, y, x)B$

2. $(x,y,z)B, (x,z,y)B \Rightarrow y = x$

3. $(x,y,z)B, (x,z,u)B \Rightarrow (x,y,u)B, (y,z,u)B$

4. $(x,y,z)B, (x,p,y)B(y,q,r)B \Rightarrow (p,y,q)B$

5. $(x,y,z)B \Leftrightarrow d(x,y)d(y,z) = 0$

*Proof.* is obvious. □

**Theorem 1.4.** *Let $(S,B,d)$ be a B-metric space.Then $S$ is said to be convex (or B-convex) $\Leftrightarrow$ given $x,y \in S$ and $a < d(x,y)$ there exists $z \in [x,y]$ such that $d(x,z) = a$*

*Proof.* Suppose $(S,B,d)$ is convex.Let $x,y \in S, a \in B$ with $a < d(x,y)$.Then there exists a unique $u \in S$ such that $(x,y,z)B$ and $d(u,x) = a'd(x,y)$.Since $d(x,y) = d(x,u) + d(u,y)$ and $d(x,u)d(u,y) = 0$,$d(u,y) = d(x,y)d'(x,u) = ad(x,y)$.If the unique u is denoted by xay then $d(xay,y) = ad(x,y), d(xay,x) = a'd(x,y)$ for all $x,y \in S, a \in B$ □

**Lemma 1.1.** *If $x,y,z \in S, a \in B$ then $d(xay,z) = ad(x,z) + a'd(y,z)$*

*Proof.* $d(xay,z) \leq d(xay,x) + d(x,z) = a'd(x,y) + d(x,z)$ and hence $d(xay,z) \leq ad(x,y) \leq ad(x,xay)(xay,z)$ so that $ad(xay,z) = ad(x,z)$. Similarly $a'd(xay,z) = ad(y,z)$ and hence d(xay,z) = ad(x,z)+a$(y,z)$ □

*Commputing the distance between the sides using the lemma above, we get*

**Theorem 1.5.** *For all $x,y,v \in S, a,b \in B, (xay)b(uav) = (xby)a(ybv)$*

**Lemma 1.2.** *Let $S$ be a convex B-metrc space and $x_1, x_2....x_n \in S$ and $\{a_1, a_2....a_n\}$ is a partition of $B$. Then there exists a unique $x \in S$ such that $\sum a_i\alpha(x_i,u)$ for all $x \in S$*

*Proof.* Proof is clear for $n = 1$ and true for $n = 2$ by lemma in 9.1.2(vi).Assume the lemma for $n > 1$ so that there exists a $y \in S$ such that $d(y,u) = \sum a_i d(x_i,u) + (a_{n-1} + a_n)d(x_{n-1},u)$.Now $d(x,u) = a_n d(x_n,u) + a'd(y,x) = \sum a_i d(u_i,x_i)$. So such x exists and now $d(x,x_n) = \sum a_i d(x_i,x_n)$ for each $i = 1,2....n = \sum a_i d(x_i,x_k)$ for $i \neq k < \sum a'_i = a'_k$
Hence if u,v both satisfy the last statement in the lemma that $d(u,v) < d(x,x_k) + d(v,x_k a'_k)$ for all x and so $d(x,v) < \prod a'_i = 0$. Hence $u = v$ □

**Theorem 1.6.** *Let $S$ be any B-metric space. Then $S$ can be isometrically embedded in a convex B-metric space $\overline{S}$ such that if $S$ can be isometrically embedded in a convex B-metric space then $S$ can be isometrically embedded in $T$ (In other words every B-metric space has a convex hull)*

*Proof.* Let $S^*$ bounded B-extension of S.i.e., $S^*$ is the family of all mappings $f : S \longrightarrow B$ satisfying the conditions

1. $f(x)f(y) = 0$ for $x \neq y$
2. f(x)=0 except for a finite number of elements x
3. $\sum_{x \in S} f(x) = 1$

Define $\rho : S^* \times S^* \longrightarrow B$ by $\rho(f,g) = \sum f(x)g(y)d(x,y)$.It can be easily seen that $\rho$ is Boolean pseudometric (i.e.,$\rho$ satisfies $(m_1),(m_2),(m_3)$).It may not satisfy $(m_4) : \rho(f,g) = 0 \Rightarrow f = g$.Now define $f \sim g \Leftrightarrow \rho(f,g) = 0$. $\sim$ defines an equivalence relation on $S^*$.Now let $\overline{S} = S/\sim$(the quotient set).Let $f$ denote the equivalence class representing f ( for each $f \in S^*, \overline{f} \in \overline{S}$ ). Clearly if we put $\overline{\rho}(\overline{f},\overline{g}) = \rho(f,g)$ then $\overline{\rho}$ is well defined and $\overline{\rho} : \overline{S} \times \overline{S} \longrightarrow B$ is a B-metric.

If $f,g \in S^*$ and $a \in B$ then define $fag : S \longrightarrow B$ by $(fag)(x) = af(x) + a'g(x)$ then $fag \in S^*$ and $\overline{fag} = \overline{f}a\overline{g}$.Hence follows that $\overline{S}$ is a convex B-metric space.

If $s \in S$ define $s^* \in S$ by

$$s^* = \begin{cases} 1 & if x = s \\ 0 & if x \neq s \end{cases} \quad (1.0.2)$$

and then $\rho(s^*,t^*) = \sum_{x,y \in S} s^*(x), s^*(y)\rho(x,y) = \rho(s,t)$ for all $s,t \in S$ and hence the mapping $\sigma : S \longrightarrow \overline{S}$ defined by $\sigma(s) = s^*$ is a congruent embedding.

Now suppose that T is a convex B-metric space and $\alpha : S \longrightarrow T$ be a congruent embedding. If $f \in S^*$ then $\{f(x)|)x \in S\}$ is a partition of B and hence there exists a unique $z \in T$ such that

$$\rho(z,u) = \sum_{x \in S} f(x)(\rho(f(x),x) \text{ for all } x \in S$$

If the above z is denoted by $\alpha^*(f)$ then the above equation can be written in the form
$\rho(\alpha^*(f),u) = \sum_{x \in S} f(x)\rho(\alpha(x),u)$ for all $u \in T, x \in S$.Hence if $f,g \in S^*$ then $\rho(\alpha^*(f),\alpha^*(g)) = \sum_{x \in S} f(x)\rho(\alpha^*(g),\alpha(x))$
$= \sum_{x \in S} f(x) \sum_{y \in S} g(y)\rho(\alpha(y),\alpha(x))$
$= \sum_{x,y \in S} f(x)g(y)\rho(\alpha(x),\alpha(y))$
$= \sum_{x,y \in S} f(x)g(y)\rho(x,y)$ for $x,y \in S$ (Since $\alpha$ is a congruent embedding)
$= \rho(f,g) = \overline{\rho}(\overline{f},\overline{g})$.Hence if $\overline{\alpha} : \overline{S} \longrightarrow \overline{T}$ is defined by $\overline{\alpha}(\overline{f}) = \alpha^*(f,g)$then $\overline{\alpha}$ is a congruent embedding. Finally if $x \in S$ then $(\overline{\alpha} \circ \sigma)(x) = \alpha^*(\overline{x})$ and $\rho(\alpha^*(x^*,u) = \sum_{y \in S} x^*(y)f(x)\rho(\alpha(y),x)$ for all $u \in T$.Hence $(\overline{\alpha} \circ \sigma)(x) = \alpha(x)$
i.e., $\overline{\alpha} \circ \sigma = \alpha$ □

**Remark 1.1.** *This notion of convexity defers completely fromthat of Blumenthal[],Penning[]etc,even in the case of B-metric spaces. Observe that B is always B-convex in this sense (introduced here be Subrahmanyam) where as B is convex in the sense if and only if B is atom free.*

# Chapter 10

# Boolean Vector Spaces

## 1 Definitions and Examples

**Definition 1.1.** *Let $(V,+,0)$ be an abelian group and $(B,+,\cdot,',0,1)$ be a Boolean algebra.$V$ is said to be a Boolean vector space over $B$ if and only if there exists a mapping $\cdot : B\times V \to V$ denoted by $(a,x)\mapsto a\cdot x$ such that*

*(i)* $a(x+y) = ax + ay \forall x,y \in V, a \in B$

*(ii)* $a.(b.x)x = a.(b.x) \forall a,b \in B, x \in V$

*(iii)* $1x = x \ \forall x \in V$

*(iv)* $(a+b)x = a.x + b.x$ *if* $ab = 0$ *for all* $a,b \in B$ *and* $\forall x \in V$.

**Example 1.1.** *Let $(B,+,\cdot,0,1)$ be a Boolean Algebra and let $V = (B,*,0)$ be the additive group of the corresponding Boolean Ring. Here $a*b = ab' + a'b$*

**Example 1.2.** *Let $R$ be any ring $(R,+.\cdot,0,1)$ with unity and $B$ be the set of central idempotent of $R$. Then $B = (B,\vee,\wedge,',0,1)$ is a Boolean Algebra where $a\vee b = a+b-ab, a\wedge b = a.b$ and $a' = 1-a$.Let $(V,+,0,-)$ be the additive group of the ring $R$ and for $a \in B, x \in V$ define $a.x$ as the ring product of $a$ and $x$. Then $V$ is a Boolean vector space over $B$.*

*We have the following as an obvious*

**Example 1.3.** *Let $B = \{0,1\}$ be the two element Boolean Algebra and $(V,+,0,-)$ be any abelian group. Define $0.x = 0, 1.x = x$ for any $x \in V$. Then $V$ is Boolean vector space over $B$.*

**Example 1.4.** *Let $B = \wp(S)$,the power set of $S$. Let $V = G^S$ i.e., the set of all functions from $S$ into $G$, where $G$ is any abelian group. $V = G^S = \{f | f : S \to G, f is a mapping\}$.For $x,y \in V$ define $(x+y)(p) = x(p) + y(p)$ for all $p \in S$. For $a \in B, x \in V$ define*

$$(a.x)(p) = \begin{cases} x(p) & if x \in G \\ 0 & if x \notin G \end{cases} \quad (1.0.1)$$

*Then V is Boolean vector space over B.*

The following theorem shows that the concepts of a B-vector space over a Boolean Algebra B and a B-module over the corresponding Boolean ring of B are completely different.

**Theorem 1.1.** *Let $(V, +, 0, -)$ be a B-vector space over a Boolean algebra $(B, +, \cdot, 0, 1)$ and $R = (B, \oplus, \cdot, 0, 1)$ to be the corresponding Boolean Ring (with $R = B, a \oplus b = ab' + a'b$ and $a \cdot b$ is same as ab in B).Then V is an R-module if and only if $x + x = 0$ for all $x \in V$.*

**Note.** First we observe that $0x = 0$ and $a0 = 0$. Now $0x = (0 + 0)x = 0x + 0x \Rightarrow 0x = 0$ (by (iv) of definition) $a0 = a(0+0) = a0+a0 \Rightarrow a0 = 0$. Next we show that $a(-x) = -(ax)$ for all $a \in B, x \in V$. $0 = a0 = a(x+-x) = ax + a(-x) \Rightarrow ax = -ax$.Next we observe that for $a, b \in B, x \in V$ $a' = x - ax$ and $(s + b)x = ax + bx - (ab)x$.
We have $x = 1x = (a + a')x = ax + a'x$(by (iv) of definition)$\Rightarrow a'(x) = x - ax$
Note that $a + b = ab' + b$ and $(ab'b = 0$.
Now $(a + b)x = (ab' + b)x = (ab')x + bx = a(x - bx) + bx = ax + bx - (ab)x$
**Proof of the theorem.** Let $a, b \in R, x \in V$. Let $\oplus$ denote the addition in R.$(a)x = (ab'+a'b)x = ax+bx-2abx$.
Consequently, V is a R-module $\Leftrightarrow 2abx = 0 \Leftrightarrow x + x = 0$
Now we introduce the concept of a norm in a Boolean vector space.

**Definition 1.2.** *A Boolean vector space V over a Boolean Algebra B is said to be B-normed (or simply normed) if and only if there exists a mapping $|| : V \to B$ ($|x|$is called norm of x) such that*

*i.* $|x| = 0 \Leftrightarrow x = 0$

*ii.* $|ax| = a|x|$

**Example 1.5.** *The additive groups of the Boolean Ring correspong to any Boolean Algebra is a normed Boolean vector space with $|x| = x$ for any x (in the additive group ie., in B)*

**Example 1.6.** *Let R be a commutative regular ring with unity (i.e., in R to each x there is an element $a \in R$ such that $xax = x$. Note that this a is not unique but xa is unique and idempotent Now let $(V, +, 0, -)$ be the additive group of the commutative regular ring $(R, +, \cdot, 0, -, 1)$ with unity and let B be the set of all idempotents of R with $\vee, \wedge, '$ are defined (as in example (ii) of 10.1.2). Then v is a Boolean vector space over the Boolean algebra $(B, \vee, \wedge, ', 0, 1$. V is normed if we define $|x| = xa$*

**Example 1.7.** *Let S be any set. $B, V, G$ etc be as in example (iv) of 10.1.2. For any $f \in V$ (i.e., f is a mapping from S into G define $|f| = \{x \in s | f^{-1}(x) \neq 0\}$. Then V is a B-normed space.*

**Theorem 1.2.** *For a Boolean vector space V over B the following statements are equivalent*

*i. V is B-normed*

*ii. To each $x \in V$ corresponds an element $a_x \in B$ such that (1) $a_x x = x$, (2) If $b \in B$ and $bx = x$ then $a_x leq b$*

*Assume (i).Let $x \in V$. Take $a_x = |x|$. Then $|x - a_x x| = |a'_x x| = a'|x| = a'a_x = 0 \Rightarrow a_x x = x$*
*Let $b \in B$ and $bx = x \Rightarrow a_x = |x| = |bx| = b|x| = ba_x \Rightarrow a_x \leq b$.Thus (i)$\Rightarrow$ (ii).Now assume (ii). First observe that $cx = 0 \Rightarrow ca'$ for any $c \in B$ because $cx = 0 \Rightarrow c'x = x \Rightarrow a_x \leq c' \Rightarrow ca_x \leq ca' = 0$ Hence $b \in B$ and $b(cx) = cx$then $b'cx = 0 \Rightarrow b'ca_x = 0 \Rightarrow ca_x \leq b$.Hence $(ca_x)x = cx$ so that $a_{cx} = ca_x$.Set $|x| = a_x$. Then this defines a norm.*

**Corollary 1.1.** *Let $V$ be B-normed. Then for any $x, y \in V, |x + y| \leq |x| + |y|$*

*Proof.* We have $(|x| + |y|)(x + y) = |x|(x + y) + |y|(x + y) - |x||y|(x + y) = x + y \Rightarrow |x + y| \leq |x| + |y|$ □

**Corollary 1.2.** *If $V$ is a N-normed vector space then $d(x, y) = |x - y|$ defines a metric on $V$*

*Proof.* is clear. □

*Note that a normed B-vector space $V$ is a OMM space(in fact metric OMM space $(V, B, d)$ where $(B, \vee, 0)$ acts as additive monoid ).*

**Definition 1.3.** *(i) Let $V$ be a B-vector space. A finite set of nonzero elements $x_1, x_2, ....x_n$ of $V$ is called (Linearly)independent over $B$ if and only $a_1x_1+a_2x_2+....a_nx_n = 0$ and $a_1a_2....a_n \neq 0$ implies $x_1+x_2+....+x_n = 0$. A subset $S$ of $V$ of non zero elements is said to be linearly independent iff every finite subset of $S$ is linearly independent over $B$. (ii) A basis of $V$ is linearly independent subset of $V$ which spans $V$.*

**Example 1.8.** *Let $B$ be the Boolean Algebra of all subsets of a set $S$ and let $V$ be the set of all mappings of $S$ into finite subsets of an abelian group $G$. Then $V$ becomes a Boolean vector space over $B$ as follows: $f : S \to A_1, G : S \to A_2$ are in $V$, define $f + g$by $(f + g)(x) = f(x) + g(x)$ for any $x \in S$.Clearly $f + g$ is a mapping defined into a finite subset of $G$.*
*For all $a \in B, f \in V$ define $af \in V$ by*

$$(af)(x) = \begin{cases} f(x) & if x \in G \\ 0 & if x \notin G \end{cases} \quad (1.0.2)$$

*If $K$ is the set of all nonzero constant maps in $V$ then $K$ is a basis for $V$.*

# 2 Bounded B-extension of a group

**Definition 2.1.** *Let $B$ be any Boolean Algebra and let $G$ be any abelian group. A partition $B$ over $G$ is a mapping $xG \to B$ such that $x(g) = 0$ for all but a finite number of elements of $g \in G$ and $x(g)x(h) = 0$ for $g \neq h$ and $\sum_g x(g) = 1$*

*The set of all such partitions of $B$ over $G$ is called a Bounded B-extension of $G$ and will be denoted by $V$. It can be seen that $V$ is a normed B-vector space where by definition*

*1* $(x + y)(g) = \sum_{h+k=g} x(h)y(k)$

*2* $|x| = \sum_{g \neq 0} x(g)$ *and*

*3*

$$(ax)(g) = \begin{cases} ax(g)f(x) & if g \neq 0 \\ a|x|' & otherwise \end{cases} \quad (2.0.1)$$

*We state below few important theorems without proof as the proofs need lengthy preparation.*

**Theorem 2.1.** *1 A Boolean vector space V over B has a basis if and only if V is isomorphic to the bounded B-extension of an abelian group.*

*2 Every finite basis of V has exactly same number of elements.The same number is called the dimension of V.*

*3 The group of isometries of V of dimension k is isomorphic to the B-extension of the symmetric group of degree k+1*

*4 Every isometry of V is a finite product of involutions.*

# 3 Isometries

*In this section we study the geometry of Boolean vector spaces (not necessarily normed). These (general) Boolean vector spaces do not come under the category of OMM spaces as there is no concept of a metric in them. But they possess a rich geometry.*

***Notation.*** *Let V be any Boolean vector space over a Boolean Algebra B. For each $x \in V$ , let $\|x\| = \{a \in B | ax = 0\}$. We call $\|x\|$ the annhilator of x.*

*We observe that $\|x\|$ is always nonempty subset of B.We now have*

**Lemma 3.1.** *$\|x\| = B \Leftrightarrow x = 0$ and $\|x\| \wedge \|y\| \leq \|x + y\|$ for each $x, y \in V$*

*Proof.* $\|0\| = B$ is clear as $a0 = 0$ for any $a \in B$. $\|x\| = B \Rightarrow 1 \in \|x\| \Rightarrow 1x = 0 \Rightarrow x = 0$.Let $a \in \|x \wedge \|y\| \Leftrightarrow ax = 0 = ay \Rightarrow a(x + y) = 0 \Rightarrow a \in \|x + y\|$ □

**Theorem 3.1.** *For a mapping $T : V \longrightarrow V$the following statements are equivalent*

1. *T is an affine transformation of V*
2. *$T(ax + a'y) = aTx + a'Ty$ for all $x, y \in V, a \in B$*
3. *$\|x - y\| \leq \|Tx - Ty\|$*

***Note.*** *A mapping $T : V \longrightarrow V$is called a linear endomorphism of $V \Leftrightarrow T(ax + by) = aTx + bTy$ for all $x, y \in V, a, b \in B$ with $ab = 0$.If T is a linear endomorphism of V then the mapping $x \mapsto Tx + x_0$ for some $x_0 \in V$ is called an affine transformation of V.*

*We need a lemma to prove the theorem.*

**Lemma 3.2.** *A mapping $f : V \longrightarrow V$ is a linear endomorphsim of $V \Leftrightarrow f(ax) = af(x)$ for all $x \in V, a \in B$*

*Proof.* Assume $f(ax) = af(x)$ for all $x \in V, \in B$. Let $x, y \in V, a, b \in B$ with $ab = 0$.Then $f(ax + by) = (a + a')f(ax+by) = af(ax+by)+a'f(ax+by) = f(ax)+f(by) = af(x)+bf(y)$.Hence f is a linear endomorphsim.The converse is obvious. □

***Proof of the theorem***

*Assume(1). Let T be given by $Tx = Sx + x_0$ where S is a linear endomorphism of V. Then if $x, y \in V, a \in B$ $T(ax + a'y) = S(ax + a'y) + (a + a')x_0 = a(Sx + x_0) + a'(Sy + x_0) = aTx + a'Ty$. Thus (1)⇒(2)*

*Now assume (2). Let $a \in \|x - y\| \Rightarrow ax = ay \Rightarrow aT(x) + a'Ty = T(ax + a'y) = T(ay + a'y) = Ty$. Similarly $aT(x) + a'Ty) = Tx$.Thus $Tx = Ty$ which implies $a \in \|Tx - Ty\|$. Thus (2)⇒(3)*

*Finally assume (3). Put $Sx = Tx - T0$.Let $x \in V, a \in B$. Then $a' \in \|ax\| \subseteq \|Tx - T0\| = \|S(ax)\|$ and hence $a'S(ax) = 0$.Also $a \in \|x - ax\| \subseteq \|Tx - T(ax)\|\|Sx - S(ax)\|$ so that $aSx = aS(ax) = S(ax)$.Hence by the above lemma, S is a linear endomorphism of V so tat T is an affine transformation.*

# 4 Betweeness and Segments

**Definition 4.1.** *For $x, y, z \in V$,we say y lies between x and z if and only if $a \in B, y = ax + a'z$.Also the segment$[x, z]$ is the set of all y such that y lies between x and z.*

***Note.*** *Observe that*

1. *$x, z \in [x, z]$ for all $x, z \in V$*
2. *If V is the Boolean vector space obtained from a ring R with uniy and B is the set of all central idempotents of R, then $y \in [x, z] \Leftrightarrow y = ax + (1 - a)z$ for some $a \in B$*

*We now have the following*

**Theorem 4.1.** 1. *Betweeness in V is invariant under all affine transformations of V.*
2. *Betweeness is invariant under all translations of V.*

*Proof.* of (1) Let $Tx = Sx + x_0$ where S is a linear transformation of V (linear endomorphism) □

*Proof.* of (2) Let $y \in [x, z] \Rightarrow y + x \in [x + u, z + u]$ for all $x \in V$ □

**Lemma 4.1.** *$y \in [x, z] \Leftrightarrow y \in [z, x]$.Also if y7 $\in [x, z]$ and $z \in [x, y]$ then $y = z$*

*Proof.* First part is obvious aa $(a')' = a$ for any $a \in B$.For the second part, as by the theorem 10.2.3(i)as betweeness is invariant under all affine transformtions, we may assume that $x = 0$.Now suppose $y \in [0, z]$ and $z \in [0, y] \Rightarrow y = az$ and $z = b'y$for some $a, b \in B$ so that $y = a'b'y = b'a'y = b'y = z$ □

**Lemma 4.2.** *$y \in [x, z]$ and $z \in [x, u] \Rightarrow y \in [x, u]$ and $z \in [y, u]$*

*Proof.* We may assume $x = 0$ and suppose $y \in [0, z]$ and $z \in [0, x] \Rightarrow y = a'z$ and $z, = b'u$ for some $a, b \in B \Rightarrow y = a'b'u = (a + b)'u$ so that $y \in [0, u]$.Also $by = ba'b'u = 0$ so that $z = by + b'u$ implying $z \in [y, u]$.Thus the proof is complete. □

***Note*** *By symmetry, it follows that the statements*

*A.* $y \in [x,z]$ *and* $z \in [x,u]$

*B.* $y \in [x,u]$ *and* $z \in [y,u]$ *are equivalent.*

**Lemma 4.3.** *If* $u, v \in [x,z]$ *then* $[u,v] \subseteq [x,z]$

*Proof.* We may assume $x = 0$.Now we have$u = a'z$ and $v = b'z$ and $t = cu+c'v$ for some $a, b, x \in B, t \in [u,v]$.Now we have $t = cu + c'v = ca'z + c'ab'z = (ca' + a'b')z \Rightarrow t \in [0,z]$ □

*Now we have the following theorem.*

**Theorem 4.2.** *Let* $x \in V$ *and write* $y <_x z \Leftrightarrow y \in [x,z]$*.Thus* $<_x$ *is a partial ordering on V(with respect to x) and for* $x, y \in V$ *the partially ordered sets* $(V, <_x)$ *and* $(V, <_y)$ *are isomorphic under a translation on V*

*Proof.* $(V, <_x)$ is a partially ordered set in view of the lemmas in 10.2.3 □

*We shall hereafter write* $<$ *for* $<_x$ *(is the partial ordering with respect to 0)*

**Definition 4.2.** *A subset* $S \subset V$ *is said to be convex* $\Leftrightarrow x, y \in S \Rightarrow [x,y] \subseteq V$*.Also if T is any subset of V, the convex hull of T is defined to be the smallest convex subset of V containing T*

*Note that every segment is convex (by lemma 10.2.3(i))*
*Now we have the following theorem.*

**Theorem 4.3.** *A subset S of V is convex* $\Leftrightarrow$ *for every partition* $a_1, a_2 ....., a_n$ *of B and every finite set* $x_1, x_2 ....., x_n$ *of S,*$a_1x_1 + a_2x_2 + .....a_nx_n \in S$ *(We have* $a_ia_j = 0$ *for every* $i \neq j$ *and* $\sum a_i = 1$*)*

*Proof.* Assume that S is convex.Let $x_1, x_2 ....., x_n \in S$ and $a_1, a_2 ....., a_n$ be partition of B. If $n = 2$ then $a_2 = a_1\prime$ and hence $a_1x_1 + a_2x_2 = a_1x_1 + a_1x_2\prime \in [x_1, x_2] \subseteq S$.Now assume the result for all partitions with n-1 elements. Since $\{a_1, a_2 ....., a_{n-1+}a_n\}$ is also a partition of B with n-1 elements , by inductive hypothesis follows that $y = a_1x_1 + a_2x_2 + ..... + a_{n-2}x_{n-2} + (a_{n-1} + a_n)x_{n-1} \in S$ and hence $a_1x_1 + a_2x_2 + .....a_nx_n = a_n + a_n\prime y \in S$. The converse part is obvious. □

**Theorem 4.4.** *Let* $S \subseteq V$*. Then the convex hull of S, denoted* $h(S)$ *is precisely the set of all finite sums* $\sum a_ix_i$ *where* $\{a_\alpha\}$ *is a (finite) partition of B and* $\{x_\alpha\}$ *is a finite subset of S.*

*Proof.* It is routine to show that $h(S)$ is convex and $S \subseteq h(S)$ and if $S \subseteq T$, T convex then $h(S) \subset T$ □

**Definition 4.3.** *Two subsets M and N of V are said to be congruent* $\Leftrightarrow$ *there is a bijection* $\sigma : M \longrightarrow N$ *such that* $\|\sigma(x) - \sigma(y)\| = \|x - y\|$ *for all* $x, y \in M$*.Any such mapping* $\sigma$ *is called a congruence between M and N.*

*Now we have the following*

**Theorem 4.5.** *Any congruence* $\sigma : M \longrightarrow N$ *can be uniquely extended to a congruence from h(M) toH(N)*

*Proof.* Let $x \in h(M)$. Then $x = \sum_{i=1}^{n} a_i x_i$ fr some partition $\{a_1, a_2....., a_n\}$ of B and $x_1, x_2....., x_n$. Put $\theta(x) = \sum_{i=1}^{n} a_i \sigma(x_i)$. Let $y = \sum_{i=1}^{n} b_j y_j)$ where $\{b_1, b_2....., b_m\}$ is a partition of B and $y_1, y_2....., y_m$. Then $a \in \|x - y\| \Leftrightarrow ax = ay \Leftrightarrow aa_i b_j x_i = aa_i b_j y_j$ for all i and j ( Since $\sigma$ is a congruence)$\Leftrightarrow aa_i b_j \sigma(x_i) = aa_i b_j \sigma(y_j) \Leftrightarrow a\theta(x) = a\theta(y) \Leftrightarrow a \in \|\theta(x) - \theta(y)\|$.It follows that $\theta$ is well defined and an extension of $\sigma$ and is a congruence from h(M) to h(N).Finally the uniqueness follows from the following lemma. □

**Lemma 4.4.** *If M and N are convex subsets of V, a bijection $\sigma : M \longrightarrow N$ is a congruence $\Leftrightarrow \sigma(ax + a'y) = a\sigma(x) + a'\sigma(y)$ for all $x, y\ M, a \in B$.*

*Proof.* First assume that $\sigma$ is a congruence and let $x, y, a \in B$.Since $a' \in \|ax - ay\| = \|(ax + a')y - y = \|\sigma(ax + a'y) - \sigma(y)$ and $a \in \|a'x - a'y\| = \|x - (ax + a'y)\| = \|\sigma(x) - \sigma(ax + a'y)\|$.It follows that $a'\sigma(ax + a'y) = a'\sigma(y)$ and $a\sigma(ax + a'y) = a\sigma(y)$ and hence we have $\sigma(ax + a'y) = a\sigma(x) + a'\sigma(y)$. The converse follows by imitating the proof in 10.2.3(iv). □

**Theorem 4.6.** *Let $x, y, u, v \in V$. Then the segments $[x, y]$ and $[u, v]$ are congruent $\Leftrightarrow \|x - y\| = \|u - v\|$*

*Proof.* If $\|x - y\| = \|u - v\|$ then the sets $\{x, y\}$ and $\{u, v\}$ are congruent. Since $[x, y], [u, v]$ are the convex hulls of these sets $\{x, y\}$ and $\{u, v\}$ respectively these segments $[x, y]$ and $[u, v]$ are congruent. Conversely assume that $\sigma : [x, y] \longrightarrow [u, v]$ is a congruence. Then $\sigma(x) = cu + c'v$ and $\sigma(y) = du + d'v$ for some $c, d \in B$ so that if $au = av$ then $a\sigma(x) = au = a(y)$ and hence $\|u - v\| \subseteq |\sigma(x) - \sigma y\| = \|x - y\|$.Hence by symmetry, we have $\|x - y\| = \sigma(x) - \sigma(y)\|$ □

*We conclude this chapter with the remark that Subrahmanyam introduced the concepts of "betweenness " and "congruences " in general Boolean vector spaces where there are not even the notions of metric and norm are absent.This betweenness concept coincides with the concept oin special cases such as Boolean Algebras,p-rings with unity studied by Blumenthal,Zemmer etc. This and the further study of geometry can be found in the work of Boolean vector spaces (I,II,III) by Subrahmanyam.*

*Boolean vector spaces do not come under OMM-spaces where as normed Boolean vector spaces are metric OMM-spaces.Further generalizations of Boolean vector spaces can be found in vector spaces over Regular rings (by Rajagopala Rao) and Browerian vector spaces (by K.Venkateswarlu).*

# Bibliography


[1] *G.A.Birkhoff, Lattice Theory, Amer. Math. Soc.Colloq Publications Vol XXV(1948)*

[2] *L.M.Blumenthal,Boolean geometry I, Rend-conti,del circole de, Palmermo Sereis II, Vol 1,348-361 (1952)*

[3] *Charles Holland,Intrinsic metrics for lattice ordered groups,Algebra Universalis 19(1984),142-150*

[4] *C.C.Chang,Algebraic analysis of many valued logicsTransactions of American Mathematical Society,Vol 88(1958), 467-490 (1958).*

[5] *C.C.Chang,A new proof of the completeness of the Lucascwiez Axiom,Transactions of American Mathematical Society,Vol 93(1958), 74-80(1958).*

[6] *J.G.Eliott, Autometrization and the symmetric difference, Canadian Journal of Mathematics(1953) 324-331*

[7] *D.Ellis, Autometerized Boolean algebras I ,Canadian Journal of Mathematics 83-87,3 (1951)*

[8] *D.Ellis, Autometerized Boolean algebras II ,Canadian Journal of Mathematics 145-147, (1951)*

[9] *Francaseo Paoli and Constantine Tsinakis On Birkhoff's Common abstraction Problem,Studio Logica, An international Journal of Symbolic logic, Vol 100, Nov 6, 1079-1105 (2011).*

[10] *J.Jacubic Isometries of lattice ordered groups Czechoslovak Mathematics Journal, 30(1980),142-152.*

[11] *J.Jacubic Weak Isometries of lattice ordered groups, Math.Slovoca 38(1988),133-138.*

[12] *Kamal Ranjan Roy,Newmanian Geometry-IBulletin of Calcutta Mathematical Society, 1960,52, 187-184*

[13] *Kamal Ranjan Roy,Newmanian Geometry-IIBulletin of Calcutta Mathematical Society, 1962,54, 33-40*

[14] *Milan Jasem,Weak isometries and direct decompositions of Dually Residuated lattice ordered Semi groupsMath Slovoca 1993, No.2 , 119-136.*

[15] *Newman MHA,A characterisation of Boolean lattices and rings,Journal of London Mathematical Society,16,(1941),256-272.*

[16] *E.A. Nordhaus and Leo Lapidus. Brouwerian geometry. Canad. J. Math, 6 (1954), 217-229.*

[17] *E Pitcher and MP Smiley. Transitivities of betweeness Canadian. J. Math, 6 (1954), 217-229.*

[18] *C.J.Penning. Boolean metric spaces, Ph.D., Dissertation of Missouri University, 1959.*

[19] *N.Prabhakar Rao.On certain classes of Autometrized Algebras, Ph.D., Thesis, Andhra University,1977.*

[20] *N.Prabhakar Rao, Automatnized Griss algebras , Math. Sem. Notes,Vol 5 (1977) NO 2. 1151-156.*

*[21] J. Rachunek . MV-Algebral are categorically equivalent to DRl- semigroups, Math Bohemica 123, 4, 437-441*

*[22] J. Rachunek . Polars and annhilators in representable Drl-monoids and MV- Algebras, Math Slovoca 51,1,1-12*

*[23] J. Rachunek . A duality between algebras of basic logic and Drl-monoids, Math Slovoca, 126,3, 561-569.*

*[24] Rajagopala Rao N . Vector spaces over regular rings, Math Annalen 280-291,167(1966).*

*[25] Ralph Kopperman, All topologies come from generalized metrics, City college of Newyork.*

*[26] Ramana Murthy P.V Semi Browerian algebras ,Journal of Australian Mathematical Soc. Vol.18 ,PART 3, 293-302(1974).*

*[27] P.Ranga Rao. Metric spaces with distances in a lattice ordered group ,Mathe Seminar Notes, Vol.6(1978), May 1978. XVIII,PART 3 293-302(1974).*

*[28] Subba Rao BV Autometrization of lattice ordered semigroups , Algebra and Order , Proceedings of First International symposium on ordered algebraic structures, Luminy-Marseiles 1984,349-355.*

*[29] Subba Rao BV . Representable Autometrized algebras,Ph.D., Thesis, Andhra University, 1980.*

*[30] Subba Rao BV, Phani Yedupalli. Metric spaces with distances in Representable Autometrized algebras, South east Asian Bulletin of math 42, 453-462(2018).*

*[31] Subba Rao BV and Phani Yedlapalli Geometry of A-metric spaces, South East Asian Bulletin of Math 35, 1-9.*

*[32] Subrahmanyam NV. Boolean Vector Spaces-I,Math Z, 82,422-439,1964.*

*[33] Subrahmanyam NV. Boolean Vector Spaces-II,Math , 87,401-419,1965.*

*[34] Subrahmanyam NV. Boolean Vector Spaces-III,Math Z, 100, 295-313,1967.*

*[35] Subrahmanyam NV. Convexification of Boolean metric spaces ,Calcutta Mathematical Society,DJCV 45-46,1983.*

*[36] Swaminathan V Geometry of Boolean like Like Rings ,Math Seminar notes Vol.9,(1981).*

*[37] Swamy, K. L. N., Dually residuated lattice ordered semigroups II, Math. Annalen,. 160,64-71 (1965).*

*[38] Swamy,K.L.N.Isometries in lattice ordered groups,Algebra Universalis, 8,1978,59-64.*

*[39] Swamy, K. L. N. Intrinsic metrization,South East Asian Bulletin of Mathematics, 2003, 27, 1-7.*

*[40] Swamy, K. L. N,Ranga Rao P and Venkateswarlu K. A note on Unique Intrinsic metric for Boolean Algebras, ,South East Asian Bulletin of Mathematics, 2003, 27, 352-355.*

*[41] Swamy, K. L. N., Isometries in Autometrized lattice ordered groups II, Math. Seminar Notes, July 1977.*

*[42] Swamy, K. L. N. and B.V.Subba Rao . Isometries in Drl-Semigroups Math. Seminar Notes, Vol.8,1980.*

*[43] Swamy, K. L. N. and B.V.Subba Rao . Isometries in Drl-Semigroups publication for private circulation.*

*[44] Swamy,K.L.N.Ordered Algebra (Part I).*

*[45] Wayne B Powell.On isometries in abelian lattice ordered groups, Journal of Indian Mathematical Society, 46, 1982, 189-194.*

*[46] Swamy,K.L.N.Any Drl-Semigroup is the direct product of a commutative l-group and a Drl-semigroup with least element,Discussions Math.Algebra, Stochastic methods, 16,1996,99-105.*

*[47] Venkateswarlu K.Browerian Vector Spaces Ph.D., Thesis, Andhra University, 1992*